MBL Transition from Phenomenological Renormalization Group

Zhengyan Darius Shi
Department of Physics, Stanford University, Stanford, CA 94305
(Dated: June 21, 2020)

Submitted as coursework for PH470, Stanford University, Spring 2020

In this article, we review the pioneering phenomenological renormalization group (RG) scheme proposed and solved in [1]. The scheme basically partitions a 1D system into interlocking thermal and insulating blocks. As we flow into the infrared, blocks with successively larger lengths are combined with their surrounding blocks. We show how to derive continuum flow equations for the RG scheme and how to solve for the RG fixed point. We then outline how various critical properties of the many-body localization (MBL) transition follow from a combination of exact and approximate expansions of the RG flows in a physical limit. These expansions reveal a two parameter phase plane for the MBL transition with a line of fixed points. Remarkably, the critical behavior falls under the Kosterlitz-Thouless universality class, contrary to previous expectations [2]. In the end, we quote some mathematical results that give hints of how to generalize this RG scheme to two dimensions and comment on possible research directions to explore in the future.

INTRODUCTION

Many body localization (MBL) is a novel dynamical phase of disordered quantum matter where the existence of an extensive number of local conserved quantities (referred to as l-bits) prevent the system from reaching local thermal equilibrium. These emergent conserved quantities are stable under perturbations of the model parameters, in contrast to the often non-local conserved quantities that appear in fine-tuned integrable models (see [3] for some examples of quantum integrable models). Furthermore, the presence of l-bits immediately imply many striking physical properties-preservation of local memories at infinite times, logarithmic grow of subregion entanglement entropy etc. (see [4] and references therein). These properties are not only of theoretical interest to the foundations of statistical mechanics, but also has important applications to the stable storage and processing of coherent quantum information. The groundbreaking work of Imbrie in 2014 identified these l-bits in an open subset of the parameter space (labeled by three disorder strength variables h, γ, J) of a disordered Ising model, thus putting one-dimensional MBL on a firm theoretical footing [5]. One natural question is: what happens in the rest of the parameter space? Numerically, the answer is clear: simulations have confirmed the existence of a thermal phase in Imbrie's model away from the MBL region. But it is an embarrassing fact that a rigorous proof of the more familiar thermal phase is still missing. From a mathematical perspective, this fact is perhaps not surprising, because Imbrie's proof of MBL fundamentally relies on the convergence of perturbation theory when one of the disorder parameters γ is small (along with some additional control over the non-perturbative resonances). On the thermal side, all parameters are comparable to each other and no perturbation theory should be trusted whatsoever. This is one of the fundamental challenges of strongly coupled many-body physics in general.

One additional challenge is that the transition between MBL and thermal phases is fundamentally a transition between systems in and out of equilibrium. Therefore, none of the conventional intuitions about equilibrium phase transitions applies and progress must come from novel phenomenological models. In this review, we will present a heuristic renormalization group (RG) model first introduced by Goremykina et al. in [1], the latest of a series of RG models proposed in the past five years. The earliest model [2] is microscopically motivated but inaccessible analytically. A variation [6] in 2016 obtained more precise analytic results after making an unphysical assumption about the entanglement times of thermal and MBL systems. The work we review finds a sweet spot in between [2] and [6], restoring some portion of the microscopic structure in [2] while preserving solvability. We now proceed to introduce the rules of the RG flow and analyze its properties.

RULES OF THE GAME

The general philosophy of our approach is as follows: numerically, the phenomenon of MBL is ubiquitous across models with different microscopic interactions. The theoretical challenge is not to solve all of these models precisely, but rather to identify universal features. The universal feature we focus on in this article

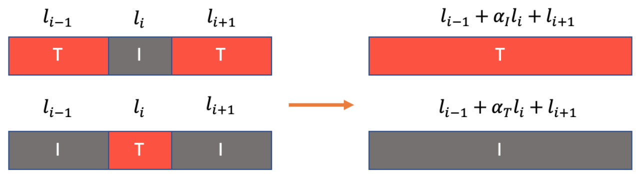

FIG. 1: The one dimensional RG rules of [1] are shown here. l_i labels the original lengths of the blocks and T/I stands for thermal/insulating blocks respectively. The new length of the combined block is a weighted sum of the lengths of the original blocks with weightings α_T, α_I chosen to reflect the entanglement times of the combined block.

is the logarithmic growth of subsystem entanglement entropy in MBL systems to be contrasted with the power law growth in thermal systems. Taking advantage of this distinction, we can partition a general 1D system unambiguously into spatially alternating thermal and insulating (localized) blocks. During the renormalization step at length scale Γ, we merge all insulating (thermal) blocks with length $l \in [\Gamma, \Gamma+\delta\Gamma]$ with its adjacent thermal (insulating) blocks as shown in Fig. 1. This $3 \to 1$ merging process always creates a new block with effective lengths satisfying the following RG rules:

$$\tilde{l}^{I,T} = l^{I,T}_{n-1} + \alpha_{T,I} l^{T,I}_n + l^{I,T}_{n+1} \tag{1}$$

where $\alpha_{I,T}$ are asymmetric scaling parameters associated with decimated insulating/thermal blocks. A priori, the presence of $\alpha_{T,I}$ may appear to be superfluous. Why don't we simply add the lengths? The intuition is that the effective length should really encode the entanglement properties of the thermal/insulating blocks. Recall that in insulating/thermal blocks, the entanglement time (i.e. the time it takes for a block's entanglement entropy with an artificial thermal bath to saturate) τ scales exponentially/linearly with the block lengths. When an insulating block is absorbed by two thermal blocks, the insulating block contributes an exponentially large entanglement time to the combined thermal block. Therefore the effective length of the combined thermal block must be much larger than the sum of lengths of the constituent blocks. This is why we expect $\alpha_I \gg 1$ (by a similar argument, we also have $\alpha_T \ll 1$). Based on this simple rule, we can derive RG flow equations for the density profiles of thermal and insulating blocks in exact analogy with the Fisher strong disorder RG in [7]. Denote by $n^{I,T}_\Gamma(l)$ the number of insulating/thermal blocks of length l at the RG length scale Γ. The density profiles are naturally defined as $\rho^{I,T}_\Gamma(l) = n^{I,T}_\Gamma(l)/N^{I,T}_\Gamma$ where $N^{I,T}_\Gamma$ is the total number of insulating/thermal blocks at RG scale Γ. From (1) we know that each microscopic step always kills one thermal block and one insulating block. Therefore, the equality $N^I_\Gamma = N^T_\Gamma$ is preserved under RG evolution. For simplicity of notations we simply denote the quantity by N_Γ.

Now observe that the RG rules in (1) imply the following equation for $n^{I,T}_\Gamma$ as we do an infinitesimal RG from $\Gamma \to \Gamma+\delta\Gamma$:

$$\begin{aligned} &n^{I,T}_{\Gamma+\delta\Gamma}(l) = n^{I,T}_\Gamma(l) + n^{T,I}_\Gamma(\Gamma)\delta\Gamma \\ &\cdot\big(-2\rho^{I,T}_\Gamma(l) + \int_\Gamma^\infty \rho^{I,T}_\Gamma(l_1)\rho^{I,T}_\Gamma(l-l_1-\alpha_{T,I}\Gamma)dl_1\big) \end{aligned} \tag{2}$$

where the first term on the RHS is the number of blocks of length l in the previous RG step, the first term in the bracket comes from eliminating the two adjacent blocks and the second term in the bracket comes from creating a new block, with the length constraint satisfied. At first glance, this equation seems incomplete. For example, in the evolution of n^I_Γ, one would have thought that we need to add an additional term $-n^I_\Gamma(\Gamma)\delta\Gamma$ that eliminates all the insulating blocks within $l \in [\Gamma, \Gamma+\delta\Gamma]$. However, this additional term is in fact not necessary because $n^I_{\Gamma+\delta}(l)$ only makes sense for $l \geq \Gamma+\delta\Gamma$ and the additional term doesn't affect $l \geq \Gamma+\delta\Gamma$.

To simplify these equations, we introduce a new variable $\zeta = l-\Gamma$ which always runs from $0 \sim \infty$. Define new variables $\rho(\zeta;\Gamma) = \rho_\Gamma(\zeta+\Gamma)$. Substituting this new definition into the flow equation, we find

$$\begin{aligned} &\rho^{I,T}(\zeta-\delta\Gamma;\Gamma+\delta\Gamma)N_{\Gamma+\delta\Gamma} - \rho^{I,T}(\zeta;\Gamma)N_\Gamma \\ &= \rho^{T,I}(0;\Gamma)N_\Gamma\delta\Gamma\big(-2\rho^{I,T}(\zeta;\Gamma) \\ &+ \int_0^\infty d\zeta_1\rho^{I,T}(\zeta_1;\Gamma)\rho^{I,T}(\zeta-\zeta_1-(1-\alpha_{T,I})\Gamma;\Gamma)\big) \end{aligned} \tag{3}$$

Plugging in $N_{\Gamma+\delta\Gamma} = N_\Gamma(1-[\rho^I(0;\Gamma)+\rho^T(0;\Gamma)]\delta\Gamma)$ and working to first order in $\delta\Gamma$, we find that

$$\begin{aligned} \frac{LHS}{N_\Gamma} &= \rho^{I,T}(\zeta-\delta\Gamma;\Gamma+\delta\Gamma)(1-[\rho^I(0;\Gamma)+\rho^T(0;\Gamma)]\delta\Gamma) \\ &\quad - \rho^{I,T}(\zeta;\Gamma) \\ &= \rho^{I,T}(\zeta-\delta\Gamma;\Gamma+\delta\Gamma) - \rho^{I,T}(\zeta-\delta\Gamma;\Gamma) \\ &+ \rho^{I,T}(\zeta-\delta\Gamma;\Gamma) - \rho^{I,T}(\zeta;\Gamma) \\ &- \rho^{I,T}(\zeta-\delta\Gamma;\Gamma+\delta\Gamma)[\rho^I(0;\Gamma)+\rho^T(0;\Gamma)]\delta\Gamma \\ &= (\frac{\partial\rho^{I,T}}{\partial\Gamma} - \frac{\partial\rho^{I,T}}{\partial\zeta})\delta\Gamma \\ &- \rho^{I,T}(\zeta;\Gamma)(\rho^I(0;\Gamma)+\rho^T(0;\Gamma))\delta\Gamma \end{aligned} \tag{4}$$

With this computation, we can now equate $\frac{LHS}{N_\Gamma\delta\Gamma}$ with $\frac{RHS}{N_\Gamma\delta\Gamma}$ in (3) and obtain a compact flow equation after suppressing the dependence on Γ in our notation

$$\begin{aligned} &\frac{\partial\rho^{I,T}(\zeta)}{\partial\Gamma} - \frac{\partial\rho^{I,T}(\zeta)}{\partial\zeta} - \rho^{I,T}(\zeta)[\rho^{I,T}(0) - \rho^{T,I}(0)] \\ &= \rho^{T,I}(0)\int_0^\infty d\zeta_1\rho^{I,T}(\zeta_1)\rho^{I,T}(\zeta-\zeta_1-(1-\alpha_{T,I})\Gamma;\Gamma) \end{aligned} \tag{5}$$

A final manipulation is to rescale the variable $\zeta \to \eta = \zeta/\Gamma$ and rescale the density profile $\rho^{I,T}(\zeta) = \frac{1}{\Gamma}Q^{I,T}_\Gamma(\eta)$.

After some simple applications of the chain rule, we obtain

$$\begin{aligned}\frac{\partial Q_\Gamma^{I,T}(\eta)}{\partial \ln \Gamma} &= \partial_\eta[(1+\eta)Q_\Gamma^{I,T}(\eta)] \\ &+ Q_\Gamma^{I,T}(\eta)[Q_\Gamma^{I,T}(0) - Q_\Gamma^{T,I}(0)] \\ &+ Q_\Gamma^{T,I}(0)\theta(\eta - \alpha_{T,I} - 1) \\ &\cdot \int_0^{\eta-\alpha_{T,I}-1} d\eta_1 Q_\Gamma^{T,I}(\eta_1) Q_\Gamma^{T,I}(\eta - \eta_1 - \alpha_{T,I} - 1)\end{aligned} \tag{6}$$

To find RG fixed points, we set the LHS to zero. This implies that we can drop the dependence of Q on Γ. Due to the presence of a convolution integral on the RHS, it is convenient to work with the Laplace transforms $\phi^{I,T}(x) = \int_0^\infty e^{-x\eta} Q^{I,T}(\eta)$. Taking Laplace transform of both sides, we get

$$\begin{aligned}0 = &\int \partial_\eta[(1+\eta)Q^{I,T}(\eta)]e^{-x\eta} \\ &+ [Q^{I,T}(0) - Q^{T,I}(0)]\int Q_\Gamma^{I,T}(\eta)e^{-x\eta} \\ &+ Q^{T,I}(0)\int_0^\infty d\eta e^{-x\eta}\theta(\eta - \alpha_{T,I} - 1) \\ &\int_0^{\eta-\alpha_{T,I}-1} Q^{I,T}(\eta_1)Q^{I,T}(\eta - \eta_1 - \alpha_{T,I} - 1)\end{aligned} \tag{7}$$

Via integration by parts, we can turn the first term into

$$-\int [(1+\eta)Q^{I,T}(\eta)](-x)e^{-x\eta} = x\phi^{I,T}(x) - x\partial_x\phi^{I,T}(x) \tag{8}$$

The second term only has a trivial η integral which turns into the Laplace transform of $Q(\eta)$. For the third term, we make a change of variables to $\eta_2 = \eta - \alpha_{T,I} - 1$. Choosing the convention that $\theta(0) = 0$, we get

$$Q^{T,I}(0)e^{-x(1+\alpha_{T,I})}\int_0^\infty d\eta_2 e^{-x\eta_2}\int_0^{\eta_2} Q^{I,T}(\eta_1)Q^{I,T}(\eta_2 - \eta_1) \tag{9}$$

Now the integral over η_1 is just a convolution integral, and the integral over η_2 is a Laplace transform of the convolution. Thus, by convolution theorem, we find

$$\text{3rd term} = Q^{T,I}(0)e^{-x(1+\alpha_{T,I})}\phi^{I,T}(x)^2 \tag{10}$$

Putting everything together, we get a differential equation for the Laplace transform

$$\begin{aligned}x\partial_x\phi^{I,T}(x) = x\phi^{I,T}(x) + [Q^{I,T}(0) - Q^{T,I}(0)]\phi^{I,T}(x) \\ + Q^{T,I}(0)e^{-x(1+\alpha_{T,I})}\phi^{I,T}(x)^2\end{aligned} \tag{11}$$

Taking derivatives on both sides with respect to x, evaluating the expression at x = 0, and then using the boundary conditions $\phi^{I,T}(0) = 1$, we obtain

$$\begin{aligned}\partial_x\phi^{I,T}(x)|_{x=0} = 1 + [Q^{I,T}(0) - Q^{T,I}(0)]\partial_x\phi^{I,T}(x)|_{x=0} \\ + Q^{T,I}(0)[2\partial_x\phi^{I,T}(x)|_{x=0} - (1+\alpha_{T,I})]\end{aligned} \tag{12}$$

To make further progress, we must assume that $Q^T(0) + Q^I(0) = 1$. This reduces the equation further

$$\partial_x\phi^{I,T}(x)|_{x=0} = 1 + \partial_x\phi^{I,T}(x)|_{x=0} - Q^{T,I}(0)(1+\alpha_{T,I}) \tag{13}$$

Eliminating the derivative from both sides, we find some constraints on $\alpha_{T,I}$:

$$Q^{T,I}(0) = \frac{1}{1+\alpha_{T,I}} \tag{14}$$

$$\frac{1}{1+\alpha_T} + \frac{1}{1+\alpha_I} = \frac{2+\alpha_I+\alpha_T}{1+\alpha_I+\alpha_T+\alpha_T\alpha_I} \tag{15}$$

We immediately conclude that consistent choices of $\alpha_{T,I}$ must satisfy $\alpha_T\alpha_I = 1$ given that $Q^T(0) + Q^I(0) = 1$. How do we justify this assumption? By examining the defining equation for N_Γ, we see that $Q^T(0) + Q^I(0) = 1$ implies an asymptotic conservation law for the total length of the 1D system:

$$\frac{dN_\Gamma}{d\Gamma} \sim -\frac{[Q^T(0) + Q^I(0)]}{\Gamma}N_\Gamma = -\frac{N_\Gamma}{\Gamma} \quad \rightarrow \quad d\frac{\Gamma N_\Gamma}{d\Gamma} = 0 \tag{16}$$

This is a pleasant feature that we naturally want to impose. But we should note that it is not as physical as we may think upon first sight: the conserved length is an effective length, not the physical length of the system. Hence there is no a priori reason why this length should be conserved under RG flow. Nevertheless, we will go ahead and assume $\alpha_T\alpha_I = 1$ since it simplifies the later analysis. All qualitative lessons we learn are not dependent on this precise relation so long as we are in the limit $\alpha_T \ll 1 \ll \alpha_I$.

EXTRACTING PHYSICS FROM THE FLOW EQUATIONS

The fundamental equations that we have derived so far are exact continuum equations of the RG rules (1). These equations are not solvable analytically for generic $\alpha_{T,I}$, but provide the starting point for various approximations and extensions that reveal physical properties:

1. For $\alpha_T \ll 1 \ll \alpha_I$, the stationary distribution $Q_*^{I,T}(\eta)$ can be solved:

$$Q_*^T(\eta) \sim \begin{cases}\frac{Q^T(0)}{(1+\eta)^{1+Q^T(0)-Q^I(0)}} & \eta \leq 1+\alpha_T^{-1} \\ e^{-\Lambda_T\eta} & \eta \geq 1+\alpha_T^{-1}\end{cases} \tag{17}$$

$$Q_*^I(\eta) \sim \begin{cases}\frac{Q^T(0)}{(1+\eta)^{1+Q^T(0)-Q^I(0)}} & \eta \leq 1+\alpha_I^{-1} \\ e^{-\Lambda_I\eta} & \eta \geq 1+\alpha_I^{-1}\end{cases} \tag{18}$$

   Since $\alpha_T \ll 1 \ll \alpha_I$, the power law region is very short in $Q_*^I(\eta)$ but parametrically long in $Q_*^T(\eta)$. This point will be important later.

Critical behavior can be extracted by expanding the distribution around the stationary solution $Q^{I,T}_{\Gamma}(\eta)$. If we assume that the dependence of the perturbation on η and Γ are decoupled, then we can write $Q^{I,T}_{\Gamma}(\eta) = Q^{I,T}_{*}(\eta) + a^{I,T}(\Gamma) f^{I,T}(\eta)$ where $a^{I}(\Gamma) = a^{T}(\Gamma) = \Gamma^{1/\nu}$. The linearized flow equation for $f^{I,T}(\eta)$ identifies ν^{-1} with the maximal eigenvalue of some integro-differential operator. An asymptotic solution of the eigenvalue problem computes ν^{-1} as a function of α_T as $\alpha_T \to 0$. This analysis is the **only step** that requires some numerical input, as there are two constants in the asymptotic solution that cannot be calculated analytically (see the supplemental materials of [1] for more details). After plugging in the power law form of $Q^{T}_{*}(\eta)$ which is valid for infinite η in the strict $\alpha_T \to 0$ limit, we obtain the asymptotic result $\nu = \ln\left(1 + \alpha_T^{-1}\right)$. The divergence of ν as $\alpha_T \to 0$ is in sharp contrast to previous finite estimates obtained in [2].

2. Since the critical exponent ν diverges as $\alpha_T \to 0$, linear perturbations are asymptotically marginal and we must go beyond linear order to understand the nature of the fixed points. A precise analysis of this kind is out of reach. What this article provides instead is an ansatz obtained by resummation of large logarithms into shifted exponents. More concretely, the authors first determine numerically that at large α_T, the eigenmodes behave like

$$f^{I}(\eta) = f^{I}_{0}(1 - I_0\eta)e^{-I_0\eta} \tag{19}$$

$$f^{T}(\eta) = f^{T}_{0}\frac{1 - \ln(1+\eta)}{(1+\eta)^2} \tag{20}$$

Plugging this expression into $Q^{T}(\eta) \approx Q^{T}_{*}(\eta) + \kappa_{\Gamma} f^{T}(\eta)$, we find that

$$Q^{T}(\eta) \approx \frac{1+\kappa_\Gamma}{(1+\eta)^2} - \kappa_\Gamma \ln(1+\eta) e^{-2\ln(1+\eta)} \tag{21}$$

The second term immediately calls for a resummation that shifts the power law exponent from $2 \to 2 + \kappa_\Gamma$. After some massaging, we see that the resummation which matches the leading order expansion obtained from the linear flow equations takes the form

$$Q^{T}(\eta) = \frac{1+\kappa_\Gamma}{(1+\eta)^{2+\kappa_\Gamma}} \tag{22}$$

A similar resummation argument applied to the series expansion $Q^{I}(\eta) = I_0 e^{-I_0\eta} + (\gamma - I_0)(1 - I_0\eta)e^{-I_0\eta}$ gives an ansatz $Q^{I}(\eta) = \gamma e^{-\gamma\eta}$. Plugging these ansatzs into the flow equations and dropping the integral terms (which are suppressed because

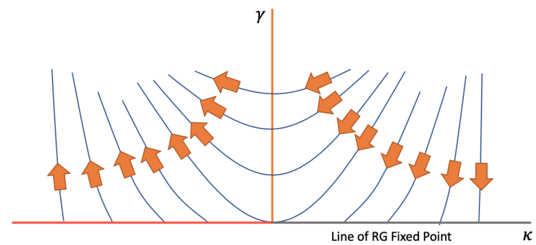

FIG. 2: The line of RG fixed points in the $\alpha_T \to 0$ limit are shown in the plot. The flow lines are meant to be schematic.

Q^{I} is exponentially suppressed, we find RG equations for the parameters γ, κ in the $\alpha_T \to 0$ limit:

$$\Gamma\frac{d\gamma}{d\Gamma} = -\gamma\kappa - \gamma^2(1+\kappa) + \gamma(\gamma\kappa + \Gamma\frac{d\gamma}{d\Gamma})\eta \tag{23}$$

$$\Gamma\frac{d\kappa}{d\Gamma}[1 - (1+\kappa)\ln(1+\eta)] = -\gamma(1+\kappa) \tag{24}$$

The authors make some further efforts towards approximating these equations. But the most important physical lesson is already transparent. Clearly, when $\gamma = 0$, the first PDE is automatically satisfied. As for the second PDE, since $\kappa, \eta > 0$, $[1 - (1+\kappa)\ln(1+\eta)] \neq 0$. This means that $\gamma = 0$ is consistent with any value of $\kappa > 0$. These observations show that there is a line of fixed points for $\gamma = 0, \kappa > 0$! In the simplified parameter space of γ, κ we thus have the schematic phase diagram as shown in Fig.2.

3. One can also take a different route and study the RG flows of variables other than the length of thermal/insulating blocks. Due to the peculiar scalings in (1), we know that when combining blocks, the total length of the new block isn't simply the sum of lengths of the old blocks. So what if we also keep track of the RG flow of total lengths of decimated blocks rather than total lengths of combined blocks? Computationally, this means that we should introduce the new RG rules

$$l^{I}_{dec} = l^{I}_{i-1} + l^{T}_{i} \quad l^{T}_{dec} = l^{T}_{i-1} + l^{T}_{i+1} \tag{25}$$

Introducing a new variable $\chi_{I,T} = l^{I,T}_{dec} - \Gamma$ in complete analogy as before, we can obtain joint flow equations for $\chi^{I,T}, l^{I,T}$. The equilibrium distributions determine a scaling of $\chi_{I,T} \sim l^{d_{I,T}}$. The exponents $d_{I,T}$ are interpreted as fractal dimensions of the insulating/thermal blocks. In the asymptotic limit $\alpha_T \to 0$, an expansion to $O(\alpha_T^3)$ can be obtained:

$$d_I = 1 - \frac{3}{4}\frac{1}{(1+\alpha_T^{-1})^2} + \frac{1}{2}\frac{1}{(1+\alpha_T^{-1})^3} \tag{26}$$

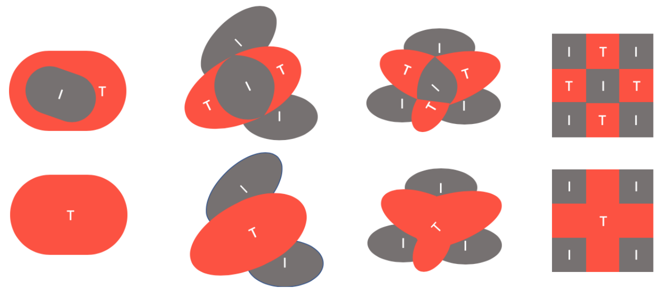

FIG. 3: RG rules in the 2D case. We show four cases where an insulating block is surrounded by 1,2,3,4 thermal blocks respectively. This is meant to demonstrate that any integer number of surrounding blocks is allowed.

$$d_T = \frac{1}{\ln\left(1+\alpha_T^{-1}\right)} \tag{27}$$

These forms suggest that $\alpha_T \to 0$ limit corresponds to $d_I \to 1$ and $d_T \to 0$. This makes sense physically because the effective lengths receive almost no contributions from the thermal blocks every step along the RG flow.

EXTENSION TO HIGHER DIMENSIONS

When we try to extend the RG schemes to higher dimensions, some challenges immediately present themselves. First of all, 1D systems have a special topology that ensures every block is always surrounded by two blocks. This allows a simple RG rule that always combines three blocks into one. The same is no longer true in higher dimensions (as we will explain in more detail later). Second of all, the natural additive quantity in higher dimensional RG is the volume of thermal/insulating blocks. However, the entanglement times scale with length rather than volume. The miracle of one dimension is that length coincides with volume. In higher dimensions, this misalignment of scales makes it hard to identify the appropriate renormalization parameters. Nevertheless, we believe that RG is at least possible in 2D due to some interesting mathematical results that relate linear and quadratic dimensions in 2D.

We consider a 2D domain of finite extent. When we partition the domain into blocks, we are giving the domain the structure of a graph. Every vertex of the graph can be assigned a degree which counts the number of blocks that intersects with the vertex. The assignment of thermal/insulating labels to these blocks is the elementary two-coloring problem. Elementary graph theory tells us that such an assignment is possible iff every vertex has even degree. Under the same assumption, we can define the RG procedure to be the combination of a central block with all blocks that share an edge with itself. This is shown pictorially in Fig.3

Remarkably, the even degree condition is preserved under the RG procedure and we get a well-defined flow. The RG rules are of course still tricky to define since we do not have a fixed number of surrounding blocks. But we can sum over all possible integer surrounding numbers and obtain a more complicated integro-differential flow equation. There is no conceptual difficulty in sight.

A more serious problem that we have to tackle is the identification of appropriate variables that keep track of changes in entanglement times under RG flow. Area is not a good candidate because a thin ribbon and a round disc can have comparable area but drastically different linear dimensions and hence entanglement times. It would be unphysical to treat them as equal objects under RG flow. We therefore need to quantify the discrepancy between area and length in an arbitrary block bounded by some curve C. This link is provided by the isoperimetric inequality in differential geometry. Since entanglement times don't care about small wiggles in the curve C, we can take all curves to be smooth. Define $L(C)$ to be the length of the curve C and $A(C)$ the area bounded by the curve. In 2D, the isoperimetric inequality says that $L^2(C) \geq 4\pi A(C)$, where equality is achieved by circles of arbitrary radius. Deviations from the perfectly symmetric circle can be encoded in the mean squared curvature $\Delta\kappa = \int ds(\kappa(s) - \langle\kappa\rangle)^2$ which is a coordinate invariant measure of the degree of irregularity in the curve C (here,$\langle\kappa\rangle$ is the mean curvature). Therefore, a preliminary proposal would be to work out RG rules for A with a $\Delta\kappa$-dependent compensation factor that controls the change in entanglement time upon block combination. In this review we don't have time to flesh out the details of this proposal, but it would be interesting to see whether the techniques we learn from the 1D RG flow can be adapted to extract some properties of MBLT in higher dimensions.

CONCLUSION

The RG analysis that we have reviewed gives a tantalizing picture for the MBL transition in one dimension, predicting a KT scaling qualitatively different from power law predictions in [2]. But ultimately it is a very crude toy model that cannot be derived from microscopic physics (the justification of RG rules in terms of entanglement times is heuristic at best). Given the techniques developed in [1], the time might be ripe for an attempt to derive the same KT scaling starting with the more microscopically motivated RG flow proposed in [2]. In a different direction, as we pointed out in the previous section, a generalization of RG to higher dimensions seems more tractable than attempts to obtain exact solutions in higher dimensions. Such a generalization may shed light on the stability of MBL in higher dimensions, a question that piques the curiosity of everyone working in the field.

[1] A. Goremykina, R. Vasseur, and M. Serbyn, Phys. Rev. Lett. **122**, 040601 (2019), URL https://link.aps.org/doi/10.1103/PhysRevLett.122.040601.

[2] R. Vosk, D. A. Huse, and E. Altman, Physical Review X **5** (2015), ISSN 2160-3308, URL http://dx.doi.org/10.1103/PhysRevX.5.031032.

[3] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1993).

[4] R. Nandkishore and D. A. Huse, Annual Review of Condensed Matter Physics **6**, 15 (2015), https://doi.org/10.1146/annurev-conmatphys-031214-014726, URL https://doi.org/10.1146/annurev-conmatphys-031214-014726.

[5] J. Z. Imbrie, Journal of Statistical Physics **163**, 998–1048 (2016), ISSN 1572-9613, URL http://dx.doi.org/10.1007/s10955-016-1508-x.

[6] L. Zhang, B. Zhao, T. Devakul, and D. A. Huse, Phys. Rev. B **93**, 224201 (2016), URL https://link.aps.org/doi/10.1103/PhysRevB.93.224201.

[7] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995), URL https://link.aps.org/doi/10.1103/PhysRevB.51.6411.