A note on Topology and Broken Symmetry in Floquet Systems

Jordi Montana-Lopez
Department of Physics, Stanford University, Stanford, CA 94305

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This paper is a review of work by Harper et al.[1] and includes relevant intuition for the mathematical tools used. We exhibit highly nontrivial topological properties of Floquet system that are not available in static Hamiltonian systems, review several classification methods for Floquet Topological Insulators, study the phenomenon of Many-Body Localization in Floquet systems, and describe the topologically nontrivial phases of binary drive Floquet systems: Time Crystals and Symmetry-Protected Topological Phases.

I. INTRODUCTION

Most systems that occur naturally have time-independent Hamiltonians. The common procedure is to plug in the Hamiltonian in Schrödinger's equation, obtain a basis of time-independent eigenstates and express the time-evolution of the system in terms of this basis of eigenstates. In the case of Floquet systems, the Hamiltonian is time-dependent and periodic, with period T:

$$H(t) = H(t + T)$$

This makes our procedure more complex, since the eigenstates won't be time-independent anymore and instead of a definite energy they will have a quasienergy ε that is defined up to periodicity: $\varepsilon = \varepsilon + n\frac{2\pi}{T}$ for any integer n. This, among other reasons related to the topology of the system described below, will make the study of the unitary time-evolution operator $U(t)$ more useful than only studying the Hamiltonian H. It satisfies Schrödinger's equation with $\hbar = 1$:

$$i\frac{d}{dt}U(t, t_0) = H(t)U(t, t_0) \Rightarrow \tag{1}$$

$$\Rightarrow U(t_0 + T, t_0) = \mathcal{T}e^{-i\int_{t_0}^{t_0+T} dt' H(t')} \tag{2}$$

where $U(t, t_0)$ is the unitary time-evolution operator given initial data at time t_0, for $0 \leq t_0 < T$. The choice of initial time t_0 doesn't really matter because any initial time will be related to time 0 by a unitary transformation:

$$U(t_0 + T, t_0) = U^\dagger(0, t_0)U(T, 0)U(0, t_0) \tag{3}$$

Since we are interested in the quasienergy of the system and unitary transformations preserve the quasienergy, we can safely choose $t_0 = 0$ from now on and simplify the notation to $U(t) := U(t, 0)$.

Let's find the time-dependent eigenstates. We first find the eigenstates at a specific point in time (say T) and then find the eigenstates at any other point in time by evolving these for a time t using the time-evolution operator $U(t)$:

$$U(T)|\phi_\alpha\rangle = e^{-i\varepsilon_\alpha T}|\phi_\alpha\rangle \tag{4}$$

Now we have the eigenstates ϕ_α at T so we evolve them to a later time t:

$$|\psi_\alpha(t)\rangle = U(t, 0)|\phi_\alpha\rangle \tag{5}$$

These are called Floquet eigenstates[5,6], and they are periodic:

$$|\psi_\alpha(t + T)\rangle = U(t + T)|\phi_\alpha\rangle = U(t)U(T)|\phi_\alpha\rangle = \tag{6}$$

$$= U(t)e^{-i\varepsilon_\alpha T}|\phi_\alpha(t)\rangle = e^{-i\varepsilon_\alpha T}|\psi_\alpha(t)\rangle \tag{7}$$

where we decomposed the evolution from 0 to $t + T$ into 0 to t and then t to T. Here we can see explicitly in the exponential that any choice of quasienergy with the periodicity as above gives the same result.

A more useful way to working with the unitary time-evolution operator is decomposing it into a term that tells us what happens at the end of each period and a term that tells us what happens within one period. Due to Floquet's theorem[7], we can decompose the unitary time-evolution operator as:

$$U(t) = \Phi(t)e^{-iH_F t}, \Phi(t) = \Phi(t + T) \tag{8}$$

where H_F is the Floquet Hamiltonian (different from H !), an effective Hamiltonian that will be most useful for us, since it's time independent and applying it evolves the system forward by T and is the term that captures the behavior at the end of each period. The term that deals with the evolution within a period is the unitary operator $\Phi(t)$; it is thus called "micromotion" operator.

Notice that if we plug in $t = 0$ in equation 8 we get $\Phi(0) = U(0)$, but $U(0) = U(0,0) = 1$ is just the identity, since there is no change from time 0 to time 0. Therefore, $\Phi(T) = \Phi(0) = 1$, so since $\Phi(t)$ starts and ends at identiy it is a "unitary loop". Even though it starts and ends at the identity, it can have a complicated behavior in between, and studying this behavior will be important for determining the topological characteristics of the system.

As we have seen, the periodic nature of the Hamiltonian in Floquet systems requires a more complicated treatment than we would usually need for time-independent Hamiltonians. However, this added complexity gives us new phenomenon that do not have a non-periodic counterpart: single-particle systems have nontrivial topology[2], many-body Floquet localization is achieved[9–13] and localized interacting phases[8] arise from broken symmetries or nontrivial topologies.

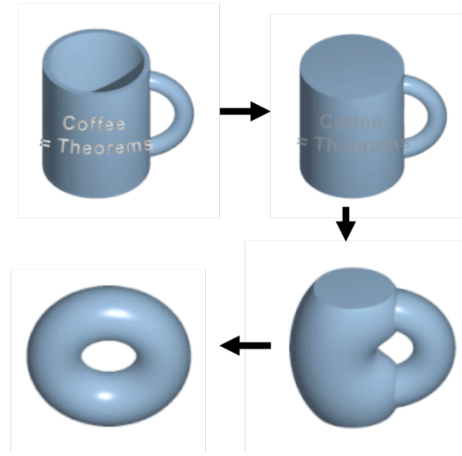

FIG. 1: Homeomorphism between a coffee mug and a donut. Modified from[4]

II. BASICS OF TOPOLOGY

Before diving into results about the topology and symmetry of Floquet systems we will have a brief review about these two topics. Feel free to skip it.

Topology is a mathematical tool that allows us to study when two objects can be deformed into another in a continuous manner. In 3D imagine we have have one 2D surfaces made of some stretchy material. We will say that they are "topologically equivalent" or "homeomorphic" if we can start with one of them and deform it (without glueing parts of it together nor piercing holes or cutting it) into the other. The common example of this is the continuous deformation between a coffee mug and a donut that we can see in figure 1. The key point is that both only have one hole, so if you deform either without glueing or piercing you will end up with some other shape that will always have a hole. Therefore, we can say that all objects with just one hole are topologically equivalent and that they all belong to the class of objects with one hole.

A topological invariant that will come in handy later is called the "winding number". It basically counts how many times a curve wounds around a point. For example, take three curves $\gamma_1(x) = (cos(x), sin(x))$, $\gamma_2(x) = (cos(2x), sin(2x))$, $\gamma_3(x) = (cos(3x), sin(3x))$. These have domain $[0, 2\pi]$, which can be thought of as a circle S^1, so we get different ways the circle S^1 describes curves on the real plane R^2. Moreover, for $x \in [0, 2\pi]$ the curve γ_1 will go around the point $(0,0)$ once, the curve γ_2 will do it twice, and the curve γ_3 will do it three times. So γ_1 has winding number 1 around the origin, γ_2 winding number 2, and γ_3 winding number 3. This would work the same for any point inside the disk $x^2 + y^2 < 1$, since the curves also wound around them the same number of times. On the other hand, if we pick a point outside of this disk, the curves are not wounding around them, so the winding number is 0.

Another concept that will be useful later is that of two Hamiltonians being "homotopically equivalent". For two functions $f, g : M \to N$, for some domain M and target space N, this just means that we can deform one continuously into the other. In general, there is a linear homotopy $h(f, g)$ given by:

$$h : M \times [0,1] \to N, h(x, s) = (1-s)f(x) + sg(x) \quad (9)$$

Notice that at $s = 0$, $h(x, 0) = f(x)$ and at $s = 1, h(x, 1) = g(x)$, and h is continuous. So one might think "there's always a homotopy between any two Hamiltonians!". The point is that we might require extra properties on those Hamiltonians, and then the domain or target space might not be the same. In that case, if we try to do a linear homotopy, the endpoint might just be outside of the target space we are working on, so it's not a valid homotopy.

III. TOPOLOGY OF SINGLE-PARTICLE FLOQUET SYSTEMS

A remarkable result in Floquet theory is that even Floquet systems with only one particle have interesting topological properties (i.e. nontrivial topology) Harper et al.[1] and references therein.

We will focus on Floquet systems on a crystal lattice, meaning there is a discrete spatial periodicity apart from the temporal periodicity of the Hamiltonian. Single-particle Floquet systems with discrete spatial translation symmetry are very similar to the case of an electron in a crystal studied by Bloch. Since Floquet systems have discrete temporal periodicity, if they also have discrete spatial periodicity, they are called "Floquet-Bloch" systems[18]. As such, many Floquet-Bloch systems exhibit phenomenon that have already been studied in the spatial crystal case (Type I Floquet systems). We focus on those Floquet systems that do not have a non-driven counterpart and exhibit behavior unique to the driven case (Type II Floquet systems). In particular, driven systems are topologically different from non-driven systems. This means that the difference is very fundamental and is not just a perturbation of a non-driven system. Moreover, it allows us to classify Floquet systems according to their topological invariants.

A. The RLBL model

As an example we will study the RLBL model described by Rudner et al.[2]. Take a system consisting of a two-dimensional 6×4 lattice with a time-dependent Hamiltonian that makes a particle hop from their current point in the lattice to a nearest neighbor: 1. To the left, 2. Upwards, 3. To the right, 4. After the 4th step, we go back to 1, since the Hamiltonian will have completed the full period T. This is illustrated in figure 2. If we start at a point in the center of the lattice, after a full period we will go back to the starting point (blue arrow). If we start at the third point in the upper edge of the lattice, diagram 1 tells us to move left, 2 tells us to move up (but we cannot, since it's the upper edge), so we do not move, 3 tells us to move to the left (we now move from the second on the top to the first), 4 tells us not to do anything (the first vertex doesn't have a downward yellow line in step 4). So on the edge, we have a leftwards movement (green arrow), and a similar reasoning gives us a rightwards movement on the bottom (red arrow). This gives us a chirality (a tendency to rotate) on the edge modes, which is one of the properties used to characterize systems. We will see that this chirality is actually a topological property intrinsic to the system.

This system follows the time-dependent Hamiltonian[2]:

$$H(t) = \sum_{\mathbf{k}} (c^\dagger_{\mathbf{k},A} c^\dagger_{\mathbf{k},B}) H(\mathbf{k}, t) \begin{pmatrix} c_{\mathbf{k},A} \\ c_{\mathbf{k},B} \end{pmatrix} \tag{10}$$

$$H(\mathbf{k}, t) = -\sum_{n=1}^{4} J_n(t)(e^{i\mathbf{b}_n \cdot \mathbf{k}}\sigma^+ + e^{-i\mathbf{b}_n\cdot\mathbf{k}}\sigma^-) + \delta_{AB}\sigma_z \tag{11}$$

where $c^\dagger_{\mathbf{k},\alpha}$ is a creation operator of a Bloch state with crystal momentum k on a sublattice $\alpha = \{A, B\}$ and J_n controls the hopping from site B to its neighbor A, $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$, where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices acting on the sublattice space, and the vectors $\{b_i\}$ are given by $b_1 = -b_3 = (a, 0), b_2 = -b_4 = (0, a)$ and a constant sublattice potential $\delta_{AB} = \varepsilon_a - \varepsilon_B$, with ε_α the energy at filled (A) or empty (B) sites. For a simplified model from Rudner et al.[2], we can choose $\delta_{AB} = 0$. The term that controls the particle-hopping tendency, J_n is some value J for the n-th direction (1.Right-left, 2.Bottom-top, 3.Left-right, 4.Top-bottom) and 0 in the other directions.

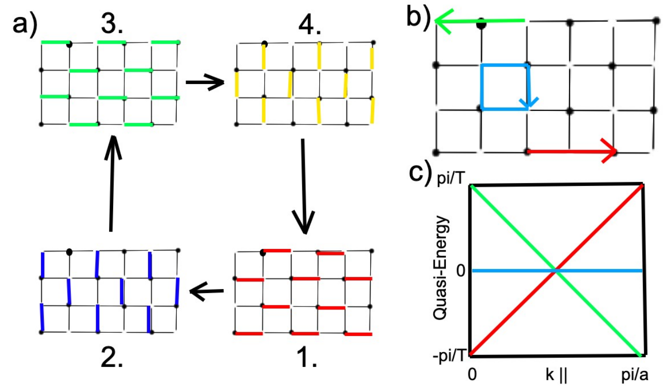

FIG. 2: 4 step lattice-hopping Floquet system. Source: J. Montana-Lopez, after Rudner et al.[2].

Figure 2 c) describes the Floquet energy spectrum in terms of the crystal momentum, of this lattice model. The red and green lines correspond to the chiral modes, which are linear in momentum ($\frac{d\varepsilon}{dk} = \pm\frac{1}{T}$)[2], while the blue line corresponds to the bulk modes, with Floquet operator the identity and quasienergy 0. We see that the difference in quasienergy in the edge lines in one period (this is called the "quasienergy gap") is high, so small perturbations will only change the diagonal lines quantitatively: maybe instead of lines they will be somewhat curved, or maybe the quasienergy gap will be a little bit smaller or larger (see the dashed lines in figure 3), but it will still have a quasienergy gap.

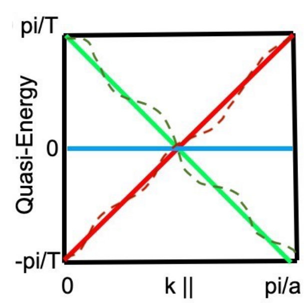

FIG. 3: Quasienergy diagram of a perturbed 4 step lattice-hopping Floquet system. Source: J. Montana-Lopez, after Rudner et al.[2].

If we consider the lattice as a whole, the edge modes are localized at different ends of the lattice, so local perturbations can not mix the edge modes to open a gap. Continuing with the analogy of the coffee mug and the donut, the chirality in the edge modes is like the hole in the donut, which will be present in any continuous deformation of the donut, and the blue loops in the bulk are like a quirky engraved inscription on the face of the coffee mug, just an accidental accessory. Therefore, chirality is a topological invariant of the system, while the loops in the bulk are not. Another way to say this is that the system has a stable "phase" with "topologically protected" edge modes. This just means that if the topology of the system is preserved then the edge modes will appear.

Chirality also gives us a reason why we really needed to study the unitary time-evolution operator: the chirality in the 4-step lattice-hopping Floquet system can not be characterized by the Floquet Hamiltonian. If we take any point not on the edge of the lattice (i.e. in the bulk, the middle), after one full period we are back to the starting point (blue line)[2]. Since the Floquet Hamiltonian only measures by timesteps of length T, if the system is the same after one period T, nothing has changed. So the unitary time-evolution operator is the identity, for periodic boundary conditions:

$$U(T) = 1 \Rightarrow H_F = 0$$

Since the Floquet Hamiltonian is time-independent, this would mean that it's zero for all times, for periodic boundary conditions. But the zero Hamiltonian doesn't make things evolve, and we just saw that the edge points have chirality. So a particle on the edge will still go around the edges and the system will be evolving, which would contradict the Floquet Hamiltonian being 0. This is because, as we mentioned earlier, the Floquet Hamiltonian deals with the behavior of the system at the end of one period, it doesn't have anything to do with what happens within one period. The evolution within a period is given by the micromotion operator $\Phi(t)$. In static system, there is a bulk-edge correspondence that allows us to find the edge properties from the invariants of the bulk (with periodic boundary conditions), but this doesn't hold in time-dependent Hamiltonians. Therefore, the Floquet Hamiltonian with periodic boundary conditions cannot capture topological behavior like chirality, so we will need new beautiful mathematical tools like K-theory and cohomology[16].

B. Classification of Floquet systems

We will be focused on Floquet Topological Insulators (FTI), which are Floquet systems on a lattice, so there is a discrete spatial symmetry. In order to better understand the kinds of FTIs that there exist it is necessary to find some general properties shared by many Floquet systems that can serve as a sort of "tag" in a classification. These tags need to be stable under small perturbations of the system so that they can group many similar systems together and we do not end up having a tag for every single problem. Topological invariants are an example of these general properties, and we will extract some from $\Phi(t)$ and some from H_F.

In the setting of a 2D translation invariant lattice with periodic boundary condition, the lattice momentum is a useful quantum number because $\Phi(t)$ will be block-diagonal in $\mathbf{k} = (k_x, k_y)$, so we can decompose $\Phi(t)$ into $\Phi(k_x, k_y, t)$, the restriction of $\Phi(t)$ to the (k_x, k_y) block, for each block. So $\Phi(k_x, k_y, t)$ is periodic in k_x, k_y, t with periods say K_x, K_y, T, so that $k_x \in [0, K_x), k_y \in [0, K_y), t \in [0, T)$. Now we can map each of these intervals to the angle of a circle, for example with $\theta_x = 2\pi \frac{k_x}{K_x}$ so that $\theta_x \in [0, 2\pi]$. If we do the same for the other three coordinates, we get a map from $S^1 \times S^1 \times S^1$ to $\Phi(k_x, k_y, t)$. As we saw earlier, different embeddings of the circle can give different winding numbers, so we can define a topological winding number for Φ. This is done in Rudner et al[2] following Bott et al.[19]:

$$\mathcal{W}[\Phi] = \frac{1}{8\pi^2} \oint dt dk_x dk_y Tr(\Phi_t^{\partial} \Phi[\Phi^\dagger \partial_{k_x} \Phi, \Phi^\dagger \partial_{k_y} \Phi]) \tag{12}$$

This is another tag used for the classification of Floquet systems because each winding number will define a "micromotion phase". In particular, nondriven systems will always have zero winding number, so it is a really powerful tool. Not only it allows us to identify if a Floquet system has nontrivial topology, but also allows us to identify whether we have found a previously unknown kind of Floquet system: look up the winding number of the Floquet systems people have already found, and if you find a system with a different winding number from those, then it's an example of a new class of Floquet systems.

We can also find topological invariants from the Floquet Hamiltonian H_F.In the case of static Hamiltonians, there is already a way to classify systems according to a topological invariant called the Chern number of the filled energy bands. The definition of the Chern number is not very elucidating for our purposes, but we can gain insight into how it's used by means of examples. In the static case, the Chern number of an energy band equals the difference between the net chirality of the edge modes traversing the gaps above and below the band[2]. The sum of the Chern number will be a topological invariant of the system. In the case of Floquet systems, the Chern number is still a topological invariant, but it can not predict the chirality of the edge modes alone.

Suppose we are given a system and we ask ourselves: will this system have chirality? For time-independent Hamiltonians, or generally for systems that reach equilibrium, there is a general framework that given the Chern number of the bands of the system tells us the absolute chirality that the system will have. However, this argument (spectral-flow) relies on the fact that

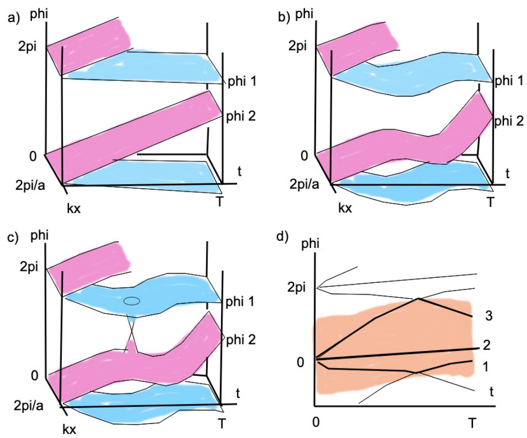

FIG. 4: Phase bands of time-evolution in single-particle Floquet systems. Source: J. Montana-Lopez, after Rudner et al.[3].

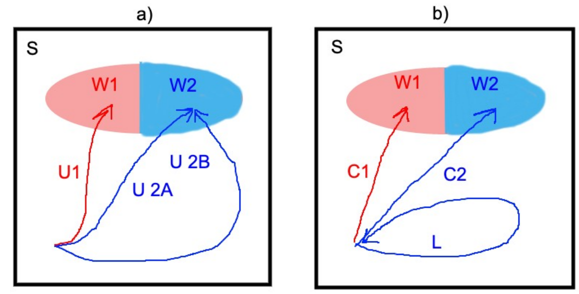

FIG. 5: Space of single-particle unitary operators with an obstruction (black spot). Source: J. Montana-Lopez, after Harper et al.[1].

there is a lowest band from which we can start counting, but in periodic systems there is no lowest band, as the spectrum is periodic in quasienergy.

In the static case, the discrete symmetries in the system are time-reversal (antiunitary), particle-hole conjugation (antiunitary) and chiral symmetry (unitary)[3]. In time-dependent systems, these symmetries will act on the Hamiltonian and the time-evolution unitary operator $U(t)$ instantaneously. Studying whether the system has these symmetries and whether the square of the antiunitary symmetries give ± 1 allows us to classify the system using the Altland-Zirnbauer (AZ) symmetry classification.

However, this classification of FTI is insufficient for Floquet systems, as we have seen that there will be other topological invariants coming from the micromotion operator $\Phi(t)$. There will also be topological invariants coming from $U(t)$ as a whole. If $U(t)$ has N energy bands and $P_n(\mathbf{k}, t)$ is the projector from $U(t)$ to its $n-$th eigenstate, then its Fourier decomposition is[3]:

$$U(\mathbf{k}, t) = \sum_{n=1}^{N} P_n(\mathbf{k}, t) e^{-i\phi_n(\mathbf{k}, t)} \tag{13}$$

and we call $\phi_n(\mathbf{k}, t) = \varepsilon_n(\mathbf{k})T$ the "phase bands". We can see examples of these phase bands in figure 4. Subfigures a) and b) are topologically equivalent, since the bands in b) can be continuously straightened out. Moreover, time-independent systems will only have linear bands, so the interesting ones for us will be those that can not be straightened out[3]. For example, subfigure c) has a topologically protected singularity because the intersection of the bands can not be straightened out, so this tells us that the system must be driven, i.e. not static.

The last way to classify FTIs uses a mathematical theory called K-theory[16]. The gist of the problem is that now we think of the Hamiltonian itself as a point in the space of possible Hamiltonians with an energy gap, in each dimension and symmetry class. Then, if we perturb this Hamiltonian a little bit we get a new Hamiltonian, meaning we move from one point to a nearby point in this space. Therefore, if we find a continuous way to deform one Hamiltonian into another, we say that they are homotopically equivalent, and they belong to the same class.

So in the static Hamiltonian case we classify the Hamiltonians based on whether we can deform them into other Hamiltonians or not. In the time-dependent case, all unitary operators are connected, so they would all be in the same class[1], which is not a great classification. But if we assume that the system doesn't have a boundary, and that there are gaps in the quasienergy at the endpoint, then we can have distinct classes of unitary operators.

Figure 5 a) shows the space of unitary operators, and we can see three different paths U_1, U_{2A}, U_{2B} with the same start and different endpoints, regions W_1, W_2, from Harper et al.[1]. These two regions can correspond to Floquet Hamiltonians with different Chern numbers (so they are topologically different in the way described above), so U_1 cannot be homotopically deformed into U_{2A}, otherwise the quasienergy gap between the two regions would close. But we have already seen that results from H_F cannot fully describe the topology of Floquet systems, so there is a more interesting result related to the micromotion. The black disk is an obstruction, so U_{2A} cannot be continuously deformed into U_{2B}, unless it is wound back to the start and then follows the path for U_{2A}, as in subfigure b). This is the classification of Floquet systems that we want: fixing H_F, what are the possible micromotions? It turns out that the classes of micromotions can be defined by loops that they would need to have to transform into another. Using K-theory[16], the equivalence classes of these loops give a way to classify the micromotions.

IV. MANY-BODY FLOQUET LOCALIZATION

Until now we have been working with single-particle systems. Adding more interacting particles will give us interesting results about localization.

For example, in a closed Many-Body Floquet system, one could expect that at long times we would achieve the state of maximal entropy, i.e. that in which the value of all local expectation values is time-independent and independent of the starting state[20]. Since all states would be available, we consider this the infinite temperature state. However, there are mechanisms to prevent this thermalization. One of them is Floquet Many-Body Localization (MBL)[9–13]. The idea is that there will be subsystems that interact with each other by receiving and transferring energy in a stable way, so that the system does not reach thermalization, but rather keeps this local exchange of energy going. The operators in these subsystems are called "l-bits"[14] and they allow for Floquet MBL systems that are stable under small perturbations.

The technique of studying the homotopies of unitary operators $U(t)$ that we just saw for single-particle systems work as well for many-body systems. Here, the regions W_1, W_2 would correspond to different (Many-body) Floquet Hamiltonians. It could be that one region only preserves one kind of symmetry, and the other region preserves another kind of symmetry, so crossing the boundary between them would mean delocalizing the system[17]. On top of that, there would be an analogous study of the obstructions and the different U_{2A}, U_{2B} as above.

Further work has shown that in driven Many-Body Ising models, four phases of matter arise. Figure 6 shows these phases in terms of two parameters of a binary drive[8]. The paramagnet and spin glass are well-known non-driven phases and the Floquet symmetry-breaking phase (SB) and Floquet symmetry-protected topological phase (FSPT) are entirely new phases of matter without a non-driven counterpart. SB systems are also known as time crystals because they break time-translation symmetry. A review on this was done by Khemani et al.[15].

V. CONCLUSION

We have reviewed several results on Floquet theory that highlight the new topological properties that Floquet phases have, and how the micromotion operator encapsules many of them. Moreover, using tools like the Winding number or the phase bands, we are now able to determine whether a system's topology is the same as that of systems with static Hamiltonians, or whether it is more complex. We learned several methods of classification for Floquet systems according to topological invariants, homotopies between Hamiltonians or unitary operators, phase bands and group cohomologies. Finally, we learned about the different phases of binary drives in Floquet systems. New directions for research include finding a complete "periodic table" of Floquet Topological Insulators, or studying how the bulk invariants are related to the invariants from the edge modes.

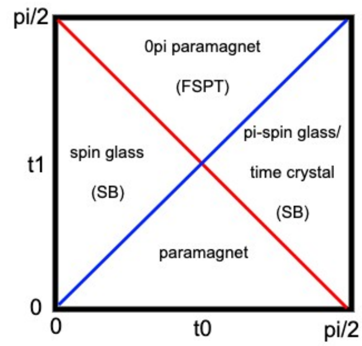

FIG. 6: Four phases of a binary driven Many-Body Ising model from[8]. Source: J. Montana-Lopez, after Khemani et al.[8].

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