Floquet Majorana end modes and topological invariants

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We study how Majorana end modes are manifested in the undriven Kitaev chain model. Zero-energy end modes appear when the bulk is in a topological phase and are protected by particle-hole symmetry and the bulk spectrum gap. The number of end modes is protected if time-reversal symmetry holds, else the parity is protected. We discuss topological invariants characterising both the cases and classify the phases of the undriven Kitaev chain according to the value of the topological invariant. We discuss Floquet theory as applied to quantum systems. We review how Majorana end modes can be induced by time-periodic driving of various parameters of the Kitaev chain even when the chain is in a trivial phase in the absence of driving. We study the construction of topological invariants that count the number of periodic driving-induced Majorana modes at Floquet eigenvalues ± 1 separately. We discuss the divergence of the number of induced end modes as drive frequency decreases. We learn that time-reversal symmetry breaking perturbations disturb the Majorana end modes and move them away from the Floquet eigenvalues ± 1. Finally, we study the effects of electron-phonon interactions and random noise in the chemical potential on the induced end modes.

I. INTRODUCTION

The use of topological invariants to identify and classify phases of matter has been an active and exciting area of research in condensed matter physics[1–3]. Typically, this technique has been applied to systems that have gaps in the bulk spectrum to understand and explain the emergence of robust zero energy modes at the boundaries. The topological invariant, generally an integer, counts the number of species of the boundary zero energy modes. The range of the topological invariant is decided by the dimension of the physical system and the symmetries it possesses. The strength and generality of the topological invariant as a classification technique arises from the fact that the invariant does not change under perturbations to the system as long as the bulk spectrum remains gapped and the symmetries are preserved.

Seminal work in the 1980s[4,5] explaining the topological origin of the quantization of conductance in the quantum Hall effect germinated this entire field and since then this technique has been applied to many other systems like two- and three-dimensional topological insulators and wires with p-wave superconductivity. In this paper, we will concern ourselves with the latter, i.e., one-dimensional p-wave superconducting wires.

Among topological systems, there has been significant excitement about many-body systems with a particular kind of zero energy boundary mode called Majorana modes. Such modes are manifest in a toy model called Kitaev chain[6] modelling a one-dimensional p-wave superconducting wire. The interest in studying Majorana end modes (MMs) was spurred on by a proposal by Kitaev and Preskill[7] outlining a way to realise quantum computing with topological qubits that are based on non-Abelian anyons and are protected against decoherence. Non-Abelian anyons can be realised in topological states of matter, specifically those that support Majorana end modes. The realization of isolated Majorana modes has been a long sought-after goal in experimental condensed matter physics.

Over time, topological classification has evolved from a technique used to provide elegant explanations for profound physics to being used as a guide to construct phases that display non-trivial and exciting physics. In the recent years, driven and out-of-equilibrium many-body quantum systems have garnered a lot of interest and, quite naturally, the topological properties of these systems have also been studied. The general problem of understanding many-body quantum systems with arbitrary time-dependence is quite difficult and remains unsolved but inroads have been made into understanding a particular class of time-dependent systems - periodic systems. Such systems are handily analysable by Floquet theory and realisable with modern experimental techniques.

Topology in Floquet systems can arise in many flavours, just like in time-independent systems[8]. For example, it has been shown that it is possible in theory to induce anomalous edge states in Chern insulators[9,10] as well as Majorana modes in otherwise topologically trivial phase of the Kitaev chain. Ref. 11 demonstrates the latter, i.e., generation of Majorana end modes by time-periodic driving of the Kitaev chain. The authors construct a novel topological invariant that counts the number of induced end-modes separately at the Floquet eigenvalues ± 1. These induced modes can persist even in the presence of electron-phonon interactions at non-zero temperature and random noise in the chemical potentials (given that these effects are sufficiently weak).

In this paper, we will review the paper[11] 'Floquet generation of Majorana end modes and topological invariants' by Thakurathi et al. which proposes using periodic driving to generate Majorana modes. Before the review, we will study the techniques used in it to build up a base that will allow us to understand the paper. In Section II, we will discuss the appearance of Majorana end modes in the undriven Kitaev chain and see how they are related to the topological invariant. In Section III, we will understand how Floquet theory is used to analyse time-periodic systems and finally, in Section IV, we will provide context for the paper being reviewed, understand what is novel about it, and dive into the details.

II. MAJORANA END MODES IN UNDRIVEN KITAEV CHAIN

This section draws heavily from the excellent collection of lecture notes[?] at https://topocondmat.org/. Readers are also encouraged to explore the papers by Matthew Radzihovsky[12] and Praveen Sriram[13] on the Physics 470 website wherein they discuss related topics.

The Kitaev chain is a tight-binding lattice model of spinless electrons with a nearest-neighbour hopping amplitude γ, a p-wave superconducting pairing Δ between neighbouring sites, and a chemical potential μ. The Hamiltonian is,

$$H = -\sum_n \mu \left(2 f_n^\dagger f_n - 1\right) + \sum_n \gamma \left(f_n^\dagger f_{n+1} + f_{n+1}^\dagger f_n\right) + \sum_n \Delta \left(f_n f_{n+1} + f_{n+1}^\dagger f_n^\dagger\right) \tag{1}$$

where f_n are annihilation operators for fermions on the nth site of the chain ($n \in \{1, \ldots, N-1\}$). The fermionic operators satisfy the usual anti-commutation relations - $\{f_m, f_n\} = 0, \{f_m, f_n^\dagger\} = \delta_{mn}$. We now construct the operators describing the Majorana modes

$$a_{2n-1} = f_n + f_n^\dagger, a_{2n} = i\left(f_n - f_n^\dagger\right) \tag{2}$$

Note that we have two Majorana operators per fermion. This holds in general - Majorana modes always appear in pairs and hence there are an even number of them and the construction procedure is reminiscent of representing a complex number as an ordered pair of two real numbers. Due to this, one might think that it is impossible to have an 'isolated' Majorana mode as physical systems are made out of electrons and they always correspond to a pair of Majoranas. We will see soon how we can be clever in designing our Kitaev chain Hamiltonian in order to 'split' a fermion and isolate two Majorana end modes at different ends of the chain. The Majorana operators satisfy $\{a_m, a_n\} = 2\delta_{mn}$ and are Hermitian, unlike the fermionic operators. In terms of the Majorana operators,

$$\begin{aligned} H &= i\sum_n \left(J_x a_{2n} a_{2n+1} - J_y a_{2n-1} a_{2n-2}\right) \\ &+ i\sum_n \mu a_{2n-1} a_{2n} \\ J_x &= \frac{1}{2}(\gamma - \Delta), J_y = \frac{1}{2}(\gamma + \Delta) \end{aligned} \tag{3}$$

From the Hamiltonian in this form, we can immediately look at some limits and draw some exciting conclusions. Consider the limit $\Delta = -\gamma, \mu = 0$. The Hamiltonian for these parameters does not feature the terms a_1 and a_{2N} and hence the chain has two zero energy modes, localized at its ends. In the other limit, $\Delta = \gamma = 0, \mu \neq 0$, we basically pair up all of the Majoranas and hence are left with no zero energy modes living at the edges of the chain. This indicates that the presence of the end modes is decided by a competition between Δ and μ and this will be reflected in the phase diagram of the system. Also, these zero modes are not just something that appear when we consider very special values for the parameters. They actually persist over a range of values. This feature is called topological protection and it's where our discussion of the topology of the Kitaev chain begins.

A. Topological protection of Majorana end modes

Topology is the study of continuous transformations. Two gapped quantum systems are called topologically equivalent if their Hamiltonians can be continuously deformed into each other without ever closing the gap. One might also want to impose more specificity; some symmetry of the system should also be preserved during the continuous transformation. A topological invariant is a quantity that does not change under continuous transformations inside the set of gapped Hamiltonians. Hence, two systems/phases are topologically equivalent if and only if their topological invariants are equal. A topological phase transition arises when the topological invariant changes and, by definition, the gap must close then.

In the context of the Kitaev chain, we first explore the symmetries of the chain. The Hamiltonian of the Kitaev chain is can be cast into the form,

$$H = \sum_{nm} f_n^\dagger A_{nm} f_m + \frac{1}{2}\left(D_{nm} f_n^\dagger f_m^\dagger + D_{nm}^* f_m f_n\right) \tag{4}$$

As f_n are fermionic operators, the matrix D is antisymmetric. Grouping the fermionic operators into a vector $F = \left(f_1, \ldots, f_n, f_1^\dagger, \ldots, f_n^\dagger\right)$, we can derive the following relations,

$$H = \frac{1}{2} F^\dagger H_{BdG} F \tag{5}$$

$$H_{BdG} = \begin{pmatrix} A & D \\ -D^* & -A^* \end{pmatrix} \quad (6)$$

We can see from this form that the Kitaev chain has a symmetry called particle-hole symmetry. The particle-hole symmetry is unlike the ubiquitous unitary symmetry that most readers may be familiar with. The symmetry operator is given by $\mathcal{P} = \tau_x \mathcal{K}$ where $\mathcal{K}$ is the complex conjugation operator (in the basis in which H_{BdG} is defined in 6) and τ_x is the block Pauli-x operator (defined on the block-structure of H_{BdG} shown in 6) and the following relation holds,

$$\mathcal{P} H_{BdG} \mathcal{P}^{-1} = -H_{BdG} \quad (7)$$

This implies that the spectrum of the Kitaev chain must be symmetric about zero. The Kitaev chain is actually equivalent to a spin-$\frac{1}{2}$ XY chain placed in a transverse magnetic field. The Jordan-Wigner transformation,

$$a_{2n-1} = \left(\prod_{j=1}^{n-1} \sigma_j^z \right) \sigma_n^x$$
$$a_{2n} = \left(\prod_{j=1}^{n-1} \sigma_j^z \right) \sigma_n^y \quad (8)$$

changes the Hamiltonian into

$$H = -\sum_n \left(J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y \right) - \sum_n \mu \sigma_n^z \quad (9)$$

The Kitaev chain is also time reversal symmetric. The time reversal symmetry operator is just complex conjugation $\mathcal{K}$ acting on the spin Hamiltonian; all objects get complex conjugated, including $i \to -i$. This also implies that

$$a_{2n} \to -a_{2n}, a_{2n+1} \to a_{2n+1} \quad (10)$$

The Kitaev chain also possesses a parity symmetry corresponding to a reflection of the system about its midpoint. This is a unitary symmetry that commutes with the Hamiltonian. Such symmetries generally don't affect the topological characteristics of the system but can make computation of the spectrum easier by reducing the dimension of the problem at hand.

With the symmetries in mind, we can now understand why the edge modes are topologically protected. Consider a pair of edge modes at zero energy. Particle-hole symmetry precludes the possibility of individually shifting one of the modes away from zero energy as the spectrum must remain symmetric about zero. Thus, the only way the zero modes can be moved away from zero is by coupling them. But this is also prohibited by their spatial separation and the presence of the energy gap in the bulk that does not allow any zero-energy excitations along the length of the wire. Hence, the only way to get rid of the MMs is to close the bulk-energy gap and that is exactly when a topological phase transition occurs. The MMs are protected by particle-hole symmetry and by the bulk gap.

B. Bulk topological invariants for Kitaev chain

Majorana end modes are intimately related to the bulk properties of the chain. This is an example of the bulk-edge correspondence principle holding; we can define topological invariants - integers that are calculated from the properties of a gapped bulk phase and characterise the presence and properties of the Majorana end modes. As pointed out before, topological classification of phases is done by differentiating them on the basis of the value of the TI.

Let us consider a more general form of the Kitaev chain (this will be useful later for understanding the effect of Floquet driving on the chain described by equation 1),

$$H = i \sum_{m,n=1}^{2N} a_m M_{mn} a_n \quad (11)$$

where M is a real antisymmetric matrix. This general Hamiltonian also has particle-hole symmetry ($i \to -i$, $a_m \to a_m$ under the action of $\mathcal{P}$) and hence a spectrum symmetric about zero. The symmetric eigenvectors are complex conjugates of each other, $iMx_j = \lambda_j x_j$ and $iMx_j^* = -\lambda_j x_j^*$. The zero eigenvalues must be even in number and their eigenvectors can be chosen to be real because $iMx_j = 0 \implies iMx_j^* = 0$. Time-reversal symmetry however is not guaranteed and only holds if $M_{mn} = 0$ whenever both m, n are even or both are odd. Like the Kitaev chain in 1 we assume that the system is discrete translation invariant and has periodic boundary conditions so that $M_{mn} = M_{(m-n) (\mathrm{mod}\ 2N),0}$. We define fermions by using 2, and Fourier transform them $f_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} f_n e^{ikn}$ where k goes from $-\pi$ to π in steps of $\frac{2\pi}{N}$. Then the Hamiltonian can be written in momentum space as

$$H = \sum_k \left(f_k^\dagger f_{-k} \right) h_k \begin{pmatrix} f_k \\ f_{-k}^\dagger \end{pmatrix}$$
$$h_k = a_{2,k} \tau^y + a_{3,k} \tau^z \quad (12)$$

where τ^y, τ^z are the 2×2 Pauli-y, Pauli-z matrices and $a_{i,k}$ are real and periodic functions of k and the corresponding dispersion is given by $E_k = \sqrt{a_{2,k}^2 + a_{3,k}^2}$. For example, for the Hamiltonian in 1,

$$a_{2,k} = 2\Delta \sin k$$
$$a_{3,k} = 2 \left(\gamma \cos k - \mu \right) \quad (13)$$

We define the first invariant in the following manner - we map h_k to a vector $V_k = a_{2,k}\hat{y} + a_{3,k}\hat{z}$ in the $y - z$ plane and define the angle $\phi_k = \tan^{-1} \left(a_{3,k}/a_{2,k} \right)$ made by the vector with respect to the z-axis. The topological invariant $W \in \mathbb{Z}$ (for a gapped phase) is the winding number of the vector V_k around the origin

$$W = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{d\phi_k}{dk} \quad (14)$$

Note that W is ill-defined for a gapless spectrum as $V_k = 0$ at some k. Also, W can only change as V_k passes through 0. This does not happen for small changes in h_k. Hence, W is a good topological invariant in the sense that it only changes if the gap closes. A phase is topological if $W \neq 0$ and will have $|W|$ zero-energy Majorana modes at each end of a long chain. If $W = 0$, the phase is topologically trivial and does not have any Majorana end modes.

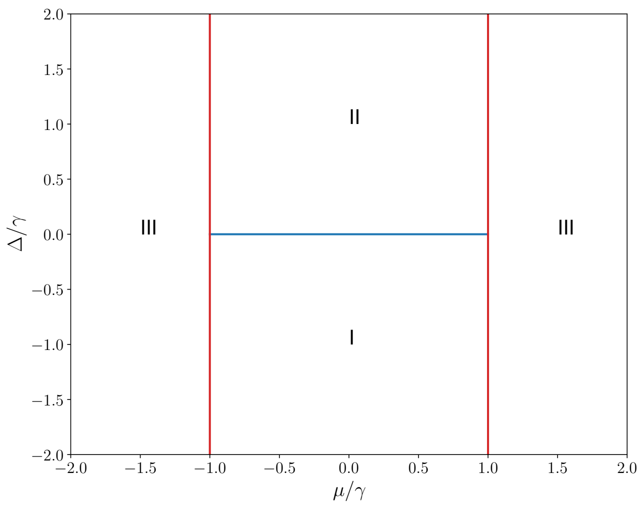

FIG. 1: Phase diagram of the undriven Kitaev chain. Adapted from Ref 11.

We can now make the phase diagram for the undriven Kitaev chain 1 and classify the phases topologically. The phase diagram is shown in Fig. 1. Considering the transformed Hamiltonian 9, in the spin-$\frac{1}{2}$ language, phase I (II) corresponds to long-range ferromagnetic order of σ^x (σ^y) and phase III corresponds to a paramagnetic phase with no long-range order. The three phases are separated by lines where the gap closes, i.e., $E_k = 0$ for some k. By taking appropriate limits (such as $\mu \ll \gamma, \Delta$ or $\mu \gg \gamma, \Delta$), we can see that W takes the values $-1, +1$, and 0 in phases I, II, and III, respectively.

In general, the TI W can take on any integer value for a general quadratic Hamiltonian 11. The Kitaev chain of equation 1 only has nearest neighbour coupling and hence the TI takes values from $\{-1, 0, 1\}$; models that have couplings with longer ranges can display phases with values beyond $\{-1, 0, 1\}$[14] and hence many pairs of MMs. We will see later that Floquet driving achieves exactly this[15] by producing effective long-range interactions while preserving time-reversal symmetry.

Also, W is only well-defined in the presence of time-reversal symmetry. When time-reversal symmetry is broken, h_k has terms proportional to $\tau^x, \mathbb{1}$[14], and the vector V_k is no longer confined to the $y-z$ plane as k goes around the Brillouin zone and a winding number can no longer be well-defined. But, it is possible to define a $\mathbb{Z}_2$-valued topological invariant ν in that case ($\nu \in \{-1, 1\}$) and this TI is called the Pfaffian invariant. It turns out that at $k = \{0, \pi\}$ h_k is always proportional to τ^z. If the spectrum is gapped such that $h_k \neq 0 \; \forall k$ and $h_0 = g_0 \tau^z$, $h_\pi = g_\pi \tau^z$, then the invariant is given by

$$\nu = \mathrm{sgn}\left(g_0 g_\pi\right) \tag{15}$$

As a time-reversal symmetry breaking perturbation is introduced, typically pairs of end modes (at each end) move away from zero energies symmetrically. Thus, in the absence of time-reversal symmetry, the number of MMs is not topologically protected but rather the parity of the number of MMs is. In the case of the original Kitaev chain that we started out with, Phase I and II have $\nu = -1$ and Phase III has $\nu = 1$.

III. FLOQUET THEORY

We will now study Floquet theory as applied to quantum systems[16,17]. Consider a time-dependent Hamiltonian with a period T, $H(t) = H(t+T)$. By Floquet's theorem, Schrödinger's equation has linearly-independent solutions of the form,

$$|\phi_j\left(t\right)\rangle = e^{-i\epsilon_j t}|\phi_j\left(t\right)\rangle \tag{16}$$

where $|\phi_j\left(t+T\right)\rangle = |\phi_j(t\rangle$ and the quasienergies ϵ_j satisfy

$$\left(H\left(t\right) - i\partial_t\right)|\phi_j\left(t\right)\rangle = \epsilon_j |\phi_j\left(t\right)\rangle \tag{17}$$

The evolution operator $U(t) = \mathcal{T}\exp\left(-i\int_0^T H\left(t\right)dt\right)$ satisfies,

$$U\left(t+T, t\right) = U\left(T, 0\right)|\phi_j\left(t\right)\rangle = e^{-i\theta_j}|\phi_j\left(t\right)\rangle \tag{18}$$

where $\theta_j = \epsilon_j T$. An effective time-independent Hamiltonian can be defined using the relation,

$$\begin{aligned} &U\left(T, 0\right) = e^{-iH_{\text{eff}}T} \\ &H_{\text{eff}}|\phi_j\left(t\right)\rangle = \epsilon_j|\phi_j\left(t\right)\rangle, \quad 0 \le t < T \end{aligned} \tag{19}$$

The quasienergy spectrum is $2\pi/T$ periodic, akin to the quasimomentum in Bloch's theorem. When a system has both discrete time and spatial translation symmetries, the Floquet states are labeled as $\{\epsilon_n\left(k\right)\}$ where n is a band index.

Because of the periodicity of the quasienergy spectrum, the particle-hole symmetry now maps $\epsilon \to (-\epsilon) \left(\mathrm{mod}\frac{2\pi}{T}\right)$. Unlike the time-independent case, there are now **two** special points that are mapped to themselves $\epsilon = 0 \to \epsilon = 0$ and $\epsilon = \frac{\pi}{T} \to \epsilon = \frac{\pi}{T}$. Particle-hole symmetry can now afford topological protection to edge states at these quasienergies if there is a gap around them. Therefore, a Floquet Kitaev chain can support two types of Majorana end modes with $e^{i\theta_j} = \pm 1$. This feature of being able to support end modes at quasi-energies other than zero is unique to periodically-driven systems and can lead to new phenomena that do not exist in undriven systems. For example, a discrete time-crystal can be realised in a $\epsilon = \frac{\pi}{T}$ phase[18].

We will now specify the general Floquet theory developed above for the case of a driven Kitaev chain. For a general time-periodic Hamiltonian as given in 11, $M(t) = M(t+T)$. In the Heisenberg picture,

$$\begin{aligned}\frac{da_n(t)}{dt} &= i[H(t), a_n(t)] \\ &= 4\sum_{n=1}^{2N} M_{mn}(t) a_n(t)\end{aligned} \tag{20}$$

If a is the column vector $(a_1, a_2, \ldots, a_{2N})$ and M is the matrix M_{mn}, the solution of the equation above is,

$$\begin{aligned}&a(t) = U(t,0)\, a(0)\,, \\ \text{where}\quad &U(t_2, t_1) = \mathcal{T}\exp\left(4\int_{t_1}^{t_2} dt M(t)\right)\end{aligned} \tag{21}$$

In this particular case, because $M(t)$ is anti-symmetric, $U(t_1, t_2)$ is real and orthogonal. The eigenvalues $e^{i\theta_j}$ of $U(T,0)$ then come in conjugate pairs (particle-hole symmetry) (if $e^{i\theta_j} \neq \pm 1$) as $U(T,0)\psi_j = e^{i\theta_j}\psi_j \implies U(T,0)\psi_j^* = e^{-i\theta_j}\psi_j^*$. For eigenvalues $e^{i\theta_j} = \pm 1$, the eigenvectors can be chosen to be real.

IV. FLOQUET DRIVING INDUCED MAJORANA END MODES

The paper[11] that we will review here discusses the generation of Majorana end modes by periodic driving of a Kitaev chain. The use of periodic driving to produce Majorana end modes was first proposed by Jiang et al.[19]. As we discussed earlier, the presence of long-range interactions and time-reversal symmetry can lead to multiple Majorana modes at each end and periodic driving can effectively produce such a situation[15]. Having multiple Majorana modes is attractive as an experiment could be designed based on tuning the number of MMs and this could lead to experimental signatures of the presence of MMs that are more robust to disorder and thermal excitations. In the work under review here, the authors study the generation of end modes under various types of Floquet drives. Floquet Kitaev chains can support MMs at $\theta_j = 0, \pi$; the authors propose novel topological invariants that count the number of Majorana end modes at $\theta_j = 0, \pi$, respectively. The authors also study the effect of breaking time-reversal symmetry on the end modes. We will now proceed with our review and, actually, we have already covered parts II and III and a portion of part IV of the paper[11] in our discussion above.

A. Finding end modes

MMs are eigenvectors of the Floquet operator $U(T,0)$ with eigenvalues ± 1 (in the infinitely long chain limit). The authors diagonalise $U(T,0)$ and calculate a quantity called the inverse participation ratio (IPR) to find eigenvectors that are localized at the ends and could be MMs. For normalized eigenvectors, $\sum_{m=1}^{2N} |\psi_j(m)|^2 = 1$, the IPR is defined as,

$$I_j = \sum_{m=1}^{2N} |\psi_j(m)|^4 \tag{22}$$

If ψ_j is extended equally over all sites then $I_j = 1/2N$ and $\lim_{N\to\infty} I_j = 0$. On the other hand, if ψ_j is localised over a distance ξ (which is of the order of the decay length of the eigenvector and remains constant as $N \to \infty$), then $|\psi_j(m)|^2 \approx 1/\xi$ over a region of length ξ and ≈ 0 elsewhere which means $I_j \approx 1/\xi$ which will remain finite as $N \to \infty$. For a sufficiently large N, I_j will distinguish between localized and extended states. Once the authors find a state for which $I_j \gg 1/2N$, they check the profile of $|\psi_j(m)|^2$ to check if it is an end state. Finally, the authors check if the form of $|\psi_j(m)|^2$and the value of IPR remains unchanged as N is increased.

As we mentioned earlier, the particle-hole symmetry of the Kitaev chain means that eigenvectors can only be moved from the eigenvalues ± 1 symmetrically by a coupling through the bulk (finite chain effect). With the eigenvectors of $U(T,0)$ this manifests in the candidate edge states with high IPR having eigenvalues $e^{\pm i\theta}$ with the deviation of the eigenvalues from ± 1 being a measure of the tunneling between the states localized at opposite edges. After finding the end modes as described above, the authors check if their wave functions are real in the limit of large N (recall that eigenvectors with eigenvalues ± 1 can be chosen to be real). The end modes are qualified as Majorana modes if they satisfy three properties: their Floquet eigenvalues equal to ± 1, they are separated by a finite gap from all other eigenvalues, they have real wave functions. We will discuss now the effects of various types of Floquet drives on the Kitaev chain.

B. Periodic δ-function kicks

The authors consider a chemical potential of the form,

$$\mu(t) = c_0 + c_1 \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{23}$$

This Hamiltonian has time-reversal symmetry: $H^*(-t) = H(t)\ \forall t$. The Floquet operator can be written as a product of two terms: an evolution with a constant chemical potential c_0 for time T followed by an evolution with a chemical potential $c_1\delta(t-T)$,

$$U(T,0) = e^{4M_1} e^{4M_0 T} \tag{24}$$

where the matrices M_i are defined in the sense of equation 11. The authors actually use an unitarily-equivalent symmetrized version,

$$U(T,0) = e^{2M_1} e^{4M_0 T} e^{2M_1} \tag{25}$$

The authors then proceed to diagonalise the Floquet operator for a choice of parameters that would imply a Kitaev chain in the trivial phase III if there was no driving. They observe (in Figures 2 and 3 of their paper) that the periodic driving produces MMs. As the period of the driving is varied the number of induced Majorana modes changes and the number of MMs with eigenvalues near ± 1 are labeled as $N_\pm$, respectively. The authors plot this in Figure 4 of their paper and notice a trend that the number of Majorana modes increases, albeit non-monotonically, as the period is increased.

The quantities $N_\pm$ are expected to be topological invariants as the MMs with eigenvalues near ± 1 are topologically protected by the particle-hole symmetry and the presence of gaps around these quasienergies; the only way to change these integers is to close the quasienergy gaps. The authors define two types of topological invariants: one that gives the total number of Majorana modes at each end of the chain and others that give the individual values of $N_\pm$.

Consider a system with periodic boundary conditions where the quasimomentum is a good quantum number. First the Hamiltonian is transformed into the (quasimomentum) Fourier basis and then the authors define a Floquet operator $U_k(T,0)$ (in a manner similar to eq. 25) for each value of the quasimomentum based on the 2×2 matrix h_k from equation 12. From this they observe that $U_k(T,0)$ can be equal to $\pm\mathbb{1}$ (and hence gapless) only if $k = 0, \pi$ and the driving frequency $\omega = 2\pi/T$ satisfies,

$$\omega = \frac{2\pi (c_0 \pm \gamma)}{n\pi - 2c_1} \tag{26}$$

for some $n \in \mathbb{Z}$. The $\pm$ signs correspond to $k = \pi$ and 0, respectively. Therefore the gap cannot close for any value of k other than $0, \pi$ and that too only happens for discrete values of ω.

For the first topological invariant, the authors define an effective Hamiltonian,

$$U_k(T,0) = e^{-ih_{\mathrm{eff},k}} \tag{27}$$

The choice of structure of Eq. 25 decides the structure of $U_k(T,0)$ which in turn leads to h_k having the following form,

$$h_{\mathrm{eff},k} = a_{2,k}\tau^y + a_{3,k}\tau^z \tag{28}$$

reminiscent of Eq. 12. Given this form, one can then compute the winding number W as described in Eq 14. As expected, this winding number counts the total number of Majorana modes at each edge of the chain. The authors plot this in Fig. 6 of their paper where they vary the driving frequency (to only have values that don't satisfy Eq. 26 in order to have a gapped phase) and compare the winding number W with the numerically calculated total number of MMs. The agreement is exact. An interesting point to note is that in the limit of $\omega \to \infty$, $U_k(T,0)$ becomes independent of k and therefore only a single point is obtained in the $(a_{2,k}, a_{3,k})$ plane. This corresponds to $W = 0$ and this is consistent with authors' observation that there is a maximum value of ω beyond which there are no MMs.

For constructing the second topological invariant, the authors start with the observation that $k = 0, \pi$ play a special role since $U_k(T,0)$ can be $\pm\mathbb{1}$ only at these quasimomenta. It can be shown that,

$$\begin{aligned} U_0(T,0) &= e^{i\pi b_0 \tau^z} \\ U_\pi(T,0) &= e^{i\pi b_\pi \tau^z} \end{aligned} \tag{29}$$

where they choose

$$\begin{aligned} b_0 &= \frac{4(c_0-\gamma)}{\omega} + \frac{2c_1}{\pi} \\ b_1 &= \frac{4(c_0+\gamma)}{\omega} + \frac{2c_1}{\pi} \end{aligned} \tag{30}$$

L_ω is defined as the line segment that goes from b_0 to b_π. Whenever an end of the line segment crosses an integer, i.e., ω satisfies Eq. 26, a topological phase transition occurs as $U_{0/\pi}(T,0) = \pm\mathbb{1}$ and the gap closes. In the limit, $\omega \to \infty$, the line collapses to a single point $2c_1/\pi$ that is not an integer by construction and no $U_k(T,0) = \pm\mathbb{1}$ meaning that there are no MMs. The second type of topological invariant is defined in the following manner - for any ω, the number of points $z = n \in \mathbb{Z}$ that lie inside the line segment is equal to the number of MMs at each end of the chain. Further, the numbers of points with n odd and even will give the numbers of end modes with Floquet eigenvalue equal to -1 and 1, respectively. The authors plot the numerical verification of this in Fig. 4 of their paper. Fig. 7 which is reproduced as Fig. 2 here shows a plot of the line segment L_ω as a function of ω.

The authors generalise this invariant for arbitrary values of the chain parameters and c_0 and non-integer values of $2c_1/\pi$. Assuming that b_i are not integers, consider all the integers lying in L_ω. Let $n_e^>$ ($n_o^>$) and $n_e^<$ ($n_o^<$), respectively, denote the numbers of even (odd) integers which are greater than and less than $2c_1/\pi$. Then, $N_+ = \mid n_e^> - n_e^< \mid$ and $N_- = \mid n_o^> - n_o^< \mid$. The winding number is given by $W = \mid n_e^> - n_e^< + n_o^> - n_o^< \mid$. Hence, W is generally not equal to the total number of modes $N_+ + N_-$. The authors verify the invariant against observations tabulated in Table I of their paper. Also, with this invariant, it is clear to see that as $\omega \to 0$, the length of the line segment diverges asymptotically and so does the number of Majorana modes. This is expected as longer periods correspond to longer range effective interactions in the chain[15] that can lead to more MMs.

For the special case of $\Delta = -\gamma$ and $c_0 = 0$, the authors provide analytical constructions of the MMs and an explicit proof that the number of MMs is indeed governed by the quantities b_0, b_π, and $2c_1/\pi$.

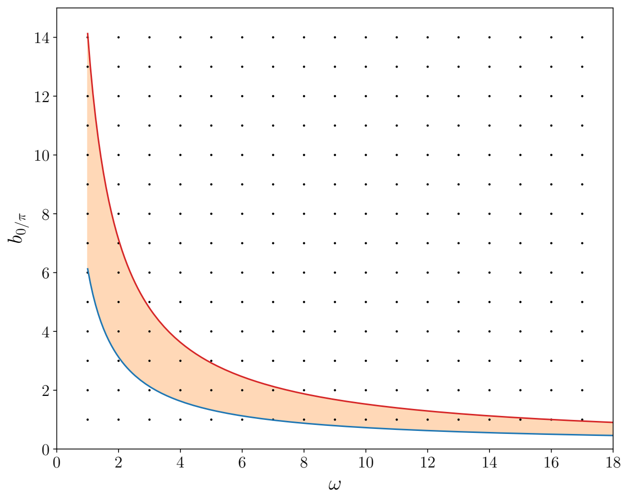

FIG. 2: Plot of b_0 and b_π as a function of ω for a system with $\gamma = 1, \Delta = -1, c_0 = 2.5$, and a periodic δ-function kick with $c_1 = 0.2$. Adapted from Ref 11.

The authors also study the effect of perturbations that break time-reversal symmetry. As discussed earlier, in this case the number of MMs is not protected as the eigenvalues move away from ± 1 symmetrically[14] but the parity of number of MMs is still protected. Indeed, the authors observe the symmetric movement of the eigenvalues away from ± 1. The deviation is small as the perturbation is small and the gap is still maintained but the eigenvectors are no longer at ± 1 and cannot be called MMs.

The authors show that periodicity in the hopping and superconducting terms can produce MMs too. In particular, they find that there is a MM at each end even in the limit of large driving frequency unlike the previous driving. The drive is,

$$\gamma = -\Delta = \gamma_0 + \gamma_1 \sum_{n=-\infty}^{\infty} \delta\left(t - nT\right)$$

By calculating $U_k\left(T, 0\right)$ just like the previous case and considering the limit of $T \to 0$, the authors find,

$$\begin{aligned} a_{2,k} &= -2\gamma_1 \sin k \\ a_{3,k} &= 2\gamma_1 \cos k \end{aligned} \tag{31}$$

which yields a winding number $W = 1$ and hence one MM even in the $T \to 0$ limit. The authors also provide an analytical proof of this fact for a special set of parameters.

C. Simple harmonic variation

The authors discuss the case where the chemical potential varies harmonically with t,

$$\mu\left(t\right) = c_0 + c_1 \cos\left(\omega t + \phi\right) \tag{32}$$

The Floquet operator is given by $U\left(T, 0\right) = \mathcal{T}\exp\left(4\int_0^T dtM\left(t\right)\right)$ same as before but in this case it does not decompose nicely into a product of only two or three operators; it has to be computed by discretising the time period and then multiplying operators in a time ordered way. Naturally, this is a more difficult numerical computation problem.

However, the authors find that the qualitative features of the MMs that arise in this case are similar to the case of periodic δ-function kicks in μ. But an interesting variable to consider here is the phase ϕ of the harmonic. The Floquet operator depends on ϕ and is labeled as $U_\phi\left(T, 0\right)$. It turns out that the eigenvalues (and hence the number of MMs) are independent of ϕ. We can see this from the fact that a shift in the phase is equal to a shift in time. Hence,

$$\begin{aligned} U_\phi\left(T, 0\right) &= U_0\left(T + \phi/\omega, \phi/\omega\right) \\ &= U_0(\left(T + \phi/\omega, T\right) U_0\left(T, \phi/\omega\right) \\ &= U_0\left(T + \phi/\omega, T\right) U_0\left(T, 0\right) U_0^{-1}\left(\phi/\omega, 0\right) \\ &= U_0\left(\phi/\omega, 0\right) U_0\left(T, 0\right) U_0^{-1}\left(\phi/\omega, 0\right) \end{aligned} \tag{33}$$

This means that $U_\phi\left(T, 0\right)$ is related to $U_0\left(T, 0\right)$ by a unitary transformation and hence they have the same eigenvalues. This also implies that studying how an eigenvector corresponding to a particular eigenvalue of $U_\phi\left(T, 0\right)$ changes with ϕ is equivalent to studying how that eigenvector changes with time under evolution with $U_0\left(\phi/\omega, 0\right)$. The effect of this evolution can be dramatic and the authors highlight this in Fig. 10 and Fig. 11 of their paper. In summary, Floquet driving induced MMs can change form with time but they remain confined to the edges and retain their quasienergies.

D. Effects of electron-phonon interactions and noise

The authors point out that experimental systems can have perturbations that do not have the same periodicity as the driving term and this can potentially affect the MMs. Consider a Majorana mode produced by driving with a frequency ω. The quasienergy gap can be defined as $\Delta E = \omega\Delta\theta/\left(2\pi\right)$ where $\Delta\theta$ is the gap between the Floquet eigenvalues of the bulk modes and the Majorana mode.

The authors assert that the MM survives if the phonon frequencies $\omega^{'}$ (mainly governed by the temperature) are much smaller than the gap ΔE and the driving frequency ω is much larger than $\omega^{'}$ and the bandwidth. Note that this implies that the large number of MMs found at small ω for the periodic δ-function kick drive in μ are not stable against electron-phonon interactions.

The authors also did numerical studies of what happens when the chemical potential has a term that is uniform in space but varies randomly with time in addition to the periodic δ-function kicks. They conclude that noise in the chemical potential does not destroy the Majorana modes if the strength of the noise is less than some value which decreases with the driving frequency ω.

V. CONCLUSION

In this paper, we started out with a discussion of the emergence of zero-energy Majorana modes in the undriven Kitaev chain and we discussed how these modes are topologically protected by the particle-hole symmetry of the chain and the presence of a bulk gap. We studied the definition of bulk topological invariants that characterise the number of Majorana modes and the phase of the chain. Next, we discussed Floquet theory for time-periodic quantum systems. Using this as a base, we studied how Floquet driving can induce many Majorana modes in a Kitaev chain that would otherwise be in a trivial phase. We understood how the authors of Ref. 11 tackled the numerical computation and search for the zero-energy end modes. We discussed the effects of various types of Floquet driving on the periodic driving-induced Majorana modes and the construction of topological invariants that characterise these modes. Breaking time-reversal symmetry resulted in the number of these modes not being topologically protected any more but rather the protection shifted to the parity of the number of modes. We also briefly discussed the effects of electron-phonon interactions and random noise on the Floquet-generated Majorana modes. Note that all our discussions here have been for a single-particle picture. Topological characteristics of Floquet systems with interactions seems to be an exciting direction to study ahead[18,20,21].

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