Critical Scaling at the MBL Phase Transition in 1D: A review of RG approaches

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This report provides an review of the recent attempts on using the renormalization group (RG) method to describe the many-body localization (MBL) transition, which describes the breakdown of thermalization behavior, and hence the conventional statistical physics description, in systems with strong disorder. In the first half of the report, we sketch out the derivation of the block decimation RG rule and explain the results obtained in the decimation approach[4–6]. We then motivate the RG rule of the toy model in Goremykina et al.[8] and demonstrate how the RG rule gives rise to the integral-differential equation that result in the final RG flow equation. We later on derive the scaling behavior of the length scale according to the RG equation. Finally, we discuss through Dumitrescu et al.[9] that touch upon results from both approaches.

I. INTRODUCTION

The ability to succinctly describe the behavior a macroscopic interacting system using only a handful of parameters lies at heart of many-body physics. While deep in thermodynamic/quantum phases it may not be too surprising that the system can be described by several macroscopic variables due to central limit theorem, formulating theories under similar paradigm in the vicinity of phase transitions can be challenging due to fluctuation at all length scales up to the macroscopic system size.

In his seminal paper[1] in 1975, Wilson conceived the scaling invariance at critical points due to the absence/divergence of length scales and put forward a elegant theory of momentum space renormalization group (RG). Applying the essence of the RG method, namely the invariance of theory under coarse graining and integrating out degrees of freedom, to real space Hamiltonians, Kadanoff constructed a powerful tool[2] to tackle the critical behavior of Ising-like systems.

The many-body localized (MBL) phase has attracted substantial interests, both theoretically and experimentally due to its potential application to quantum computing. Such phase is a class of many-body interacting systems in which, contrary to conventional wisdom from classical thermodynamics, thermalization behavior is absent and memory of the initial configuration is retained at late times. Contrary to equilibrium (ground state) phases, the MBL phase is distinct in its *dynamical* behavior[3], for instance the vanishing of DC conductance, logarithmic growth of entanglement entropy and the Poisson level statistics signifying a emergent integrability.

Characterizing the nature of the MBL-thermalization transition is both illuminating and challenging since the conventional paradigm that focuses on the absence/presence of gaps does not work; the microscopic picture of how fluctuations is present in the system across transition is also vastly different.

The application of RG to the MBL transitions, nevertheless, is not necessarily hindered by aforementioned technical challenges as it only relies on the scaling invariace of the system. In fact, several RG approaches have been proposed to study the property of the MBL transition[4–9]. While each of them differs in the RG rules which lead to different scaling behavior and falls short on microscopic justifications, there are some overall agreements among these approaches, and a convergence trend towards reaching a final consensus is apparent. This report serves as a review and comparison of these approaches.

We review the past works in a chronological order, as later works typically build upon or refer to earlier works for consistency checks. The first works that put together a RG rule and provide a scaling behavior are by Vosk et al.[4] (VHA) and Potter et al.[5] (PVP) These two independant works theorize a similar block decimation procedure and therefore result in practically identical scaling behaviors, which is discussed in Sec. II A. Later on, Dumitrescu et al. (DVP) build upon their previous work[5] and come up with a RG rule that relies less on *ad hoc* heuristics and is perhaps more microscopically justifiable, which we summarize in Sec. II B.

The previous block decimation approaches suffers from the shortcomings that no analytic tools are readily existent to analyze the RG flow of the parameters. Therefore, the scaling behaviors are only extracted from numerical simulations, which only extend up to system size of $\sim 10^6$. In light of this, several later attempts seek to bypass matrix element calculation in the previous block decimation framework and instead construct phenomenological RG flows based on minimal models that incorporate

the past intuition with the MBL transition, such as quantum avalanche[10] and the Griffith effect, which respectively depict the behavior of thermal bubbles destroying MBL phases and insulating bubbles causing sub-diffusive transport in thermal phases. These models hence abstract away the use of Hamiltonians to describe the system. Zhang et al. provide the first analytically solvable RG model under this framework by only keeping thermal/insulating block lengths as the relevant variables[7]. This method is further generalized and improved by Goremykina et al.[8] which takes into account the asymmetry between the inclusion of thermal/insulating bubble in a insulating/thermal bath. These works are summarized in Sec. III A.

Goremykina et al.'s toy model gives a surprising Kosterlitz-Thouless (KT) RG equation and indicates the divergence of length scale to qualitatively disagree with the finite-size simulation results of VHA and DVP. The recent work by Dumitrescu et al.[9] provides numerical evidences that reconciliate the results from VHA/DVP method with the predictions from KT scaling equations. A more general argument that does not rely on the toy model is also provided to arrive at the same KT RG equation. Various results from this work is finally summarized in Sec. III B.

II. BOTTOM-UP APPROACH: RENORMALIZING COUPLINGS

The essence of the block decimation approach is to combine several local degrees of freedom (for instance spin-1/2) into a block and assign it to be locally thermalizing or localized depending on entanglement rate, namely the relative strength between coupling Γ and typical energy spacing Δ of bare states within the block. Slightly different from what is done in Kadanoff's approach, no degrees of freedom are being integrate out during each RG step, one only coarse grains the information, as one can actually extract the block size from inspecting the typical energy spacing. Each RG step involves selecting some blocks to merge into a bigger block, and modifying the coupling between this block with other blocks by inspecting the properties of the constituent small blocks. The initialization of the coupling distributions between blocks of size one and the RG rules that dictate how the blocks combine is slightly different among each of the works, which we shall go in more detail below.

A. VHA and PVP

For the sake of clarity, we first outline the RG rule applied in VHA and later provide a comparison to that in PVP. We also do not attempt to go in full detail in deriving VHA or PVP's RG rule, as many of the rules for intermediate cases are not immediately justifiable and are only applied after empirical self-consistency checks. We do provide arguments for RG rules of the two extreme cases, namely the combination between two thermal blocks or two insulating blocks, since they cleanly recover the scaling behavior of entanglement spreading deep in the MBL and eigenstate thermalizing (ETH) phases.

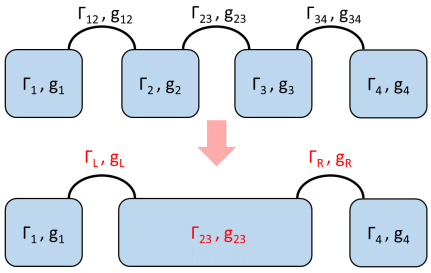

FIG. 1: The VHA model. Each block represents a finite segment of spins and is associated with a rate Γ which is proportional to the inverse thermalization time and a coupling strength g that characterizes the thermalizing/localization behavior of the block. The interblock rate Γ_{ij} is defined by the inverse time for the two blocks to fully relax. Each coarse graining step involves combining the two blocks with the strongest two-block rate together, with the new interblock rates Γ_L defined according to the RG rule (see text). For this example the two-block rate Γ_{23} is assumed to be the largest among Γ_{ij}'s, so block 2 and block 3 are decimated into a larger block in this step of RG. Figure redrawn from Vosk et al.[4]

1. *VHA RG rules*

The VHA scheme assumes a hierarchy of timescales between the one block thermalization rates Γ_i and and the interblock thermalization rates Γ_{ij}. The assumption is taken to be that a block must first thermalizes within itself before becoming entangled with other blocks.

From the scaling behavior of entanglement time deep in the two phases we can deduce the scaling of the dimensionless coupling g:

$$\begin{aligned} \textbf{ETH}: \tau \sim l, \Delta \sim 2^{-l}, g = \frac{1}{\tau\Delta} >> 1 \\ \textbf{MBL}: \tau \sim e^{-l/l_0}, l_0 \to 0, \Delta \sim 2^{-l}, g = \frac{1}{\tau\Delta} << 1, \end{aligned} \tag{1}$$

with l being the size of the system, and l_0 the localization length. Eq.1 demonstrates a clear separation of scales of the parameter g in two phases, enabling a convenient definition of the MBL phase transition to be around $g \sim 1$.

It is also noted that during the RG process, the hierarchy between the interblock rates and the intrablock rates has to be maintained, which implies that the new block rates after RG, such as Γ_L and Γ_R in Fig.1 has to be slower than $min(\Gamma_1, \Gamma_{23})$ and $min(\Gamma_{23}, \Gamma_4)$ respectively.

The rates Γ_L and Γ_R is basically the "three block rate" $\Gamma_{123}, \Gamma_{234}$, namely the inverse entanglement time of the three block system.

Keeping the hierarchy in mind, we proceed to motivate the RG rules in the VHA paper. The coupling between two blocks is assumed to be of the form

$$\hat{J}_{12} = J_{12}(A_1^\dagger A_2 + h.c.), \tag{2}$$

where A_1, A_2 are on-site operators that located at the interface between block 1 and block 2. The coupling strength J_{12} is given in the original microscopic Hamiltonian and is generally of order $W * O(1)$, with W being the single-particle disorder strength.

When joining two MBL blocks, we expect relative timescale to be $W^{-1} \ll \Delta^{-1} \ll (\Gamma_{12})^{-1}$, corresponding to a weak coupling to a continuum. In such regime, the relaxation timescale should be appropriately described by the Fermi's golden rule:

$$\Gamma_{12} \sim \sum_{f_1,f_2} |J_{12}|^2 |\langle i_1|A_1|f_1\rangle|^2 |\langle i_2|A_2|f_2\rangle|^2 \delta(E_{i_1,i_2} - E_{j_1,j_2}), \tag{3}$$

where i_n, f_n represent the initial states and the final states of block n, respectively. For the energy conservation to be matched, i_n and f_n should be located in the middle of the spectrum, and i_n should be close in terms of energy with f_n. Under this condition, it typically involves flipping $O(l)$ l-bits to get from i_n to j_n, giving the typical matrix element magnitude $|\langle i_1|A_1|f_1\rangle| \sim e^{-l/l_0}$. Therefore, the two block rate Γ_{12} scales as

$$\Gamma_{12} \sim \frac{W^2}{\Delta_{12}} e^{-2l_{12}/l_0}, \tag{4}$$

where Δ_{12} is the energy spacing of the two-block system that scales as $\Delta_{12} \sim W/2^{l_{12}}$, and l_{12} is the size of the two-block system which naturally satisfies $l_{12} = l_1 + l_2$. One can go through a similar treatment to obtain the one block rates Γ_1 and three block rates Γ_{123} as in the VHA paper, but the result can be obtained in a intuitive (though less rigorous) sense by making the observation that Γ_{12} actually only depends on its own properties and not the history of the constituent blocks. That is to say the one block rates, and hence the three block rates, can be constructed in a similar manner:

$$\begin{aligned} \Gamma_1 &\sim \frac{W^2}{\Delta_1} e^{-2l_1/l_0} \\ \Gamma_{123} &\sim \frac{W^2}{\Delta_{123}} e^{-2l_{123}/l_0}. \end{aligned} \tag{5}$$

The results in Eq.5 can be inverted to give the correct entanglement growth in the MBL phase: $l \sim S \sim \log(1/\Gamma) \to \Gamma \sim e^{-cl}$. One can also construct the RG rule used in the VHA paper:

$$\Gamma_L = \frac{\Gamma_{12}\Gamma_{23}}{\Gamma_2}. \tag{6}$$

For the opposite case of combining two thermalizing blocks, the timescale hierarchy becomes $W^{-1} \lesssim (\Gamma_{12})^{-1} \ll \Delta^{-1}$ so the Fermi's golden rule no longer capture the correct entanglement timescale in such regime. The two block rates in this regime is simply given by the Ohm's law:

$$l_{12} \propto (\Gamma_{12})^{-1} = (\Gamma_1)^{-1} + (\Gamma_2)^{-1}, \tag{7}$$

and the RG rule writes

$$(\Gamma_L)^{-1} = (\Gamma_{123})^{-1} = (\Gamma_{12})^{-1} + (\Gamma_{23})^{-1} - (\Gamma_2)^{-1}. \tag{8}$$

The above two rules are the only rules applied throught the RG process in the VHA paper. The separation of time scales is fuzzy when jointing thermalizing blocks and localized blocks, so the authors have to construct a somehow arbitrary set of rules that only justifies itself after numerical consistency checks.

2. *Initialization of the VHA scheme*

Instead of fully specifying the microscopic Hamiltonian, the authors only initialize the ensemble with energy spacings Δ and coupling constants g_{ij} drown from a prescribed probability distribution. The tuning parameter is directly obtained from the probability distribution of g, defined as $\langle \log g \rangle_0$. This approach lifts the necessity to specify the relationship between localization length as a function of disorder bandwidth, but as a drawback results in the inability to directly inspect the transition as a function of physically measurable quantity $|W - W_c|$. However, the scaling exponents of the lengthscale should be nevertheless unaltered.

3. *Results of VHA*

Two fixed points are found in the VHA paper. One corresponds to the infinite randomness point of MBL phase where g vanishes exponentially, the other corresponds to the fully thermal phase there g grows exponentially with system size. The universal scaling form is numerically shwon to be $\langle \log g \rangle(l) = \xi f(l/\xi)$, with a power-law divergence $\xi \sim 1/(g_0 - g_{0_c})^\nu$ and the critical exponent $\nu \sim 3.1$. The transition also manifests itself in terms of the transport behaviour. The scaling exponent α_{tr} in the relationship $l \sim \tau^\alpha \sim \tau_{tr}^{\alpha_{tr}}$ smoothly evolves from $\alpha_{tr} = 0.5$ signifying the diffusive transport in the ETH phase to $\alpha_{tr} = 0$ signifying the insulating behavior in the MBL phase. It is noteworthy that VHA method correctly recovered the Griffith effect, which predicts the sub-diffusive behavior on the ETH side of the transition, and the fact that the system is localized at the critical point $g_0 = g_{0_c}$.

4. Comparison to PVP

The RG laws in the PVP paper[5] of the block rates Γ_i are essentially identical to the VHA paper. The only difference is that they take a different initialization condition that incorporate the disorder bandwidth W as the tuning parameter. The system considered in the PVP paper reminiscent of a diordered Heisenberg chain:

$$H = \sum_i h_i \sigma_i^z + J(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + V \sigma_i^z \sigma_{i+1}^z, \quad (9)$$

where the on-site energy h_i is uniformly sampled within the disorder bandwidth $[-W, W]$. The Hamiltonian is first diagonalized in the single particle basis ($V = 0$), and the coupling V is later turned on to obtain the matrix element, which is of the form

$$V(x) \sim V e^{-x/x_0}, \quad (10)$$

which x_0 denotes the single particle Anderson localization length $x_0 \sim [\frac{1}{2} \ln(1 + (W/J)^2)^{-1}]$.

The scaling exponent PVP obtained after performing the RG is similar to that from the VHA paper:

$$\xi_{loc}/L = f(L/\xi), \quad (11)$$

with ξ_{loc} the length of the longest thermal block, and ξ the emergent length scale that diverges as $\xi \sim 1/|W - W_c|^\nu$ near the critical point, with $\nu \approx 3.5$.

B. DVP

Both aforementioned approaches, though self-consistent, face difficulties in determining RG rules between insulating and thermal clusters. Dumitrescu et al, build upon the previous PVP paper, created a modified RG method that is more justifiable in the intermediate regime[6].

Here we explain the setup of the DVP paper: the RG operations still focuses on the relative scale between energy spacing Δ and the coupling Γ of clusters of spin, and the initialization also come from the same disordered Heisenberg chain, as explained in Sec. II A 4. It is only the RG steps in the DVP paper that is different from the previous work. In order to circumvent the complexities associated with jointing MBL blocks with ETH blocks, the DVP RG rule only joins blocks when the resulting block is thermal. The result when RG operation terminates would be a single block when the system is in the ETH phase, and order l blocks when the system is in the MBL phase. The RG operation ensures that at any given time all the blocks are locally thermal, and therefore the coupling strengths are easy to compute:

$$\Gamma_{ij} \sim [max_{i_1 \in i, j_1 \in j} J_{ij}] \langle \alpha_i | A_{i_1} | \beta_i \rangle \langle \alpha_j | A_{j_1} | \beta_j \rangle, \quad (12)$$

where $|\alpha\rangle, |\beta\rangle$ represent eigenstates and A_n represent local operator at site n. The ETH hypothesis states $\langle \alpha_i | A_{i_1} | \beta_i \rangle \sim e^{-l s_{th}/2}$ with $s_t h$ a constant as a function of on-site Hilbert space dimension. For spin 1/2 systems $s_{th} = \ln 2$.

Despite the clean and well-motivated RG step formulation, the only observable that one has access to regarding the transition is the block entanglement entropy, which can be naturally obtained as functions of block length. Other observables, such as transport properties, are difficult to calculate. Nevertheless, the behavior of the entanglement entropy itself almost gives as much information of the transition as the previous works. The entropy is found to lie on a universal curve:

$$s(L/2)/L = f(L/\xi), \xi \sim 1/|W - W_c|^\nu, \nu \sim 3.2. \quad (13)$$

The fluctuation of the entanglement σ_s also provides another evidence that the critical point is localized, as the fluctuation peaks in the ETH phase.

III. PHENOMENOLOGICAL APPROACH AND THE KOSTERLITZ-THOULESS RG FLOW

The previous approaches, though physically justifiable, suffers from finite size limitations. Due to the form of the RG rules, the authors are not able to perform analytical calculation that extends to $l \to \infty$, and are limited to system size $\sim 10^6$, the capability of existent numeric tools.

Due to this constraint, another route to tackle the RG problem recently being pursued. The phenomenological approach seeks to avoid constructing RG rules from any microscopic models, and focuses on the analytical solvability instead. The RG rules are far from justifiable if one restricts to calculations in a *ab initio* sense, but they do capture the large scale behvior of the ETH and MBL phases.

A. Analytically solvable RG model

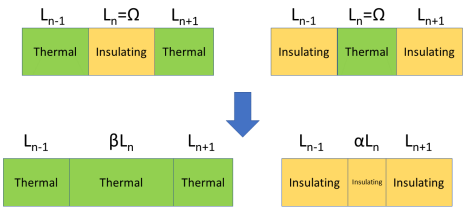

FIG. 2: The RG rule in Goremykina et al[8]. Within each RG step, blocks with length below the cutoff length scale Ω gets combined with its neighboring blocks of the opposite type into a bigger block. The difference of the "effective conductance" gives rise to the parameter α and β.

The first solvable family of RG rules is proposed by Zheng et al.[7] and expanded by Goremykina et al.[8]. In

these RG rules, the only parameters being retained are the length of the blocks, and the most simplistic Ohm's law is assumed to described the RG process:

$$\begin{aligned} l^I_{new} &= l^I_{n-1} + \alpha l^T_n + l^I_{n+1} \\ l^T_{new} &= l^T_{n-1} + \beta l^I_n + l^T_{n+1}, \end{aligned} \tag{14}$$

with a pictorical description in FIG.2.

For such artificial RG rules to emulate the actual MBL and ETH systems, the parameters describing the effective conductance α, β are set to satisfy the following conditions: $\alpha \ll 1 \ll \beta$ and $\alpha\beta = 1$. The former condition stems on the fact that localizing blocks always serve as bottleneck in terms of transport, hence the "effective resistance of a localized block is much bigger than a thermalized block. Conservation of total length during the RG process post the latter constraint.

As mentioned in FIG.2, each normalization step involves combining a block of length Ω with its neighboring blocks to form a bigger block. The cutoff length scale starts from 0 and finally reaches the total system length.

Below I extract the derivation of the RG integral-differential equation in the paper by Goremykina et al. from their supplement. Denote the number density of thermal/insulating blocks when the cutoff is $\Omega = \Gamma$ to be $\frac{dN^{I/T}}{dl} = n^{I/T}_\Gamma(l)$, and it follows that the total block number to be $N^{I/T}_\Gamma = \int_\Gamma^\infty dl n^{I/T}_\Gamma(l)$. For each step of RG, blocks of length $l \in [\Gamma, \Gamma + d\Gamma]$ are combined with their neighbors, resulting in the reduction of the total number:

$$N^{I/T}_{\Gamma+d\Gamma} = N^{I/T}_\Gamma - (n^{I/T}_\Gamma(\Gamma) + n^{T/I}_\Gamma(\Gamma))d\Gamma, \tag{15}$$

and a modification of the distribution $n^{I/T}_\Gamma(l)$:

$$\begin{aligned} n^{I/T}_{\Gamma+d\Gamma}(l) = n^{I/T}_\Gamma(l) + n^{T/I}_\Gamma(\Gamma)d\Gamma[-2\rho^{I/T}_\Gamma(l) \\ + \int_\Gamma^\infty dl' \rho^{I/T}_\Gamma(l')\rho^{I/T}_\Gamma(l - l' - \alpha/\beta\Gamma)], \end{aligned} \tag{16}$$

where the first term in the square bracket representing blocks of length l being combined with blocks of length Γ to form a longer block, and the second term is the result of generating blocks of length l after the merging process, and the probability density $\rho^{I/T}_\Gamma(l)$ is detined as $n^{I/T}_\Gamma(l)/N^{I/T}_\Gamma$.

It can then be shown that after taking the rescaled form $Q^{I/T}_\Gamma(\eta) = \Gamma\rho^{I/T}_\Gamma(l)$ of the probability distribution with $\eta = \frac{l}{\Gamma} - 1$, one arrive at the following eigenfunctions in the limit $\alpha \to 0$

$$Q^I_\Gamma(\eta) = \gamma e^{-\gamma\eta}, Q^T_\Gamma(\eta) = \frac{1+\kappa}{(1+\eta)^{2+\kappa}}. \tag{17}$$

The dynamics of the parameters η, γ gives the RG flow equation:

$$\frac{d\gamma}{d\ln\Gamma} = -\gamma\kappa, \frac{d\kappa}{d\ln\Gamma} = -\gamma(1+\kappa). \tag{18}$$

The RG flow equation can be recasted into the famous Kosterlitz-Thouless RG equation[11] that describes the 2D XY model:

$$\frac{dK^{-1}}{dl} = y^2, \frac{dy}{dl} = (2 - \pi K)y, \tag{19}$$

with K being the ratio between the spin stiffness and the temperature and y the spin vortex fugacity. The detail of the XY model is not important here, and we only note the two important features of such RG equation: First, there is a line of stable fix points. ($y = 0, K \geq 2/\pi$ in the case of KT and $\gamma = 0, \kappa \geq 0$ for the case in Eq. 2.) Second, the divergence behavior of length scales is of the form $\xi \sim e^{c/\sqrt{|P-P_c|}}$, with P being the temperature in the KT transition, and disorder bandwidth W in the MBL transition.

We here give a derivation of the scaling law from the RG flow equation Eq. 17, which is omitted in the original paper. Integrating over Eq.18 one obtains

$$\gamma_\Gamma - \kappa_\Gamma + \ln(\kappa_\gamma + 1) = C \sim C_0(W - W_c). \tag{20}$$

In the vicinity of transition, γ and κ can assumed to be small and we expand Eq.20 to lowest order of κ to obtain the relation

$$\gamma_\Gamma = \frac{\kappa^2_\Gamma}{2} + C_0(W_c - W), \tag{21}$$

and the modified RG eqution:

$$\frac{d\kappa}{d\ln\Gamma} = -\gamma_\Gamma = -(\frac{\kappa^2_\Gamma}{2} + C_0(W_c - W)) \tag{22}$$

Initially κ_Γ is of order 1 so the $C_0(W_c - W)$ term can be dropped in Eq.22 and one obtains the scaling $\Gamma \sim e^{-2/\kappa_\Gamma}$, when κ^2_Γ becomes comparable to $C_0(W_c - W)$. the flow slows down, the system approaches the fix point and the RG process is terminated. Thus, $\xi \sim e^{-2/\kappa_{\Gamma_{tr}}}$ with $\kappa^2_{\Gamma_{tr}} \sim C_0(W - W_c)$ and we obtain the KT divergence of lengthscale $\xi \sim e^{c/\sqrt{W-W_c}}$.

It is noted by the authors that such KT scaling would be hard to observe in a finite size system. There is nevertheless a scaling law that one can observe in a finite-sized system, which is the scaling exponent of thermal block probability distribution $\rho^T(l) \propto l^{-(2+\kappa)}$. Close to the critical point the fix point value κ vanishes as $\sim \sqrt{W - W_c}$, so a convergence to the $\rho^T(l) \propto l^{-2}$ could be one signature in favor of the KT RG theory.

B. Phenomenological Kosterlitz-Thouless RG flow

A later work by Dumitrescu et al.[9] removes the necessity to derive the RG flows from a somewhat artificial model. They consider the density ρ of the thermal blocks and the localization length ζ (which is not the emergent lengthscale that diverges at the transition) and write down the first order contribution to each of them

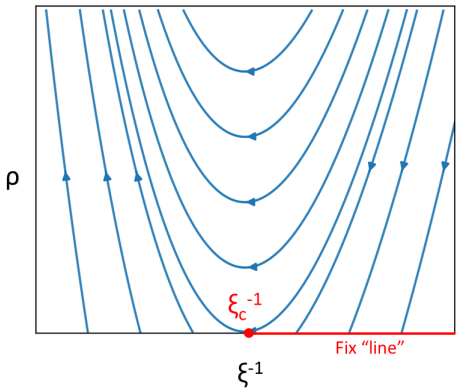

FIG. 3: The RG flow map derived in Dumitrescu et al[9]. The line of fixed points corresponds to the MBL phase. The two axes ξ and ρ represents the bare localization length and the thermalizing block density respectively. A different set of parameters is used to obtain the RG flow equation in Goremykina et al.[8], but they qualitatively describe the same physical quantities.

during the coarse graining process, which are stated below. First, the density of thermal block should grow in the ETH phase and decay in the MBL phase, which give the flow equation

$$\frac{d\rho}{dl} = b\rho(\zeta - \zeta_c) + ..., \tag{23}$$

where b is a constant of order 1. Second, the existence of thermal blocks helps extending the localization length after coarse graining, which yields the second RG flow equation

$$\frac{d\zeta}{dl} = c\rho\zeta..., \tag{24}$$

again with c being a constant of order 1. These two flow equation can be re-casted into the KT form and the resulting RG flow is plotted in FIG.3.

The authors also attempt to reconcile the KT description of the transition with the DVP/VHA RG procedure by looking at the probability distribution of block length and discover the universal scaling of $p(l) \sim l^{-2}$. Therefore, despite the fact that the bottom-up method and the phenomogical method give different predictions, these two approaches can still be compatible with each other and a refined theory that unifies both approaches awaits.

IV. CONCLUSION

We have provided an overview of several recent renormalization group approaches that studies the critical behavior of the MBL transition. The two major frameworks, bottom-up block decimation method from well-established microscopic models and phenomenological method that provides easy-to-calculate RG flow equations, though provide intuition that conforms to past experimental and theoretical finds, have their respective shortcomings. Thus, it might be necessary to combine the two frameworks to give a more comprehensive and justifiable RG rule to the MBL transition.

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