Entanglement Growth Dynamics in Condensed Matter Systems

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Entanglement is a key feature unique to quantum mechanics which ties together quantum information theory, condensed matter physics, and high energy physics. The dynamics of entanglement growth have been of much interest in recent years, being integral to the foundations of quantum statistical mechanics. In this work we summarize the important results and methods of two papers by Kim and Huse, and Nahum *et al.*. Kim and Huse study a nonintegrable quantum system which is known to thermalize by exact diagonalization a spin 1/2 chain, and find ballistic growth of the von Neumann entanglement entropy from an initially unentangled state. Nahum *et al.* instead study random unitary circuits with minimal structure, finding again generic ballistic entanglement spreading, which they map to a well-known equation governing stochastic dynamics in classical mechanics. Notably, they find that the speed of entanglement (thermalization) is in fact slower than the speed at which a single initially localized operator spreads.

I. INTRODUCTION

Entanglement is a unique feature of quantum mechanics, with no analogue in classical physics. Initially poorly understood[1], these quantum correlations that develop between different parts of a composite quantum system are fundamental to the underlying quantum evolution, and are crucial to the majority of the distinctive properties of quantum mechanics. Entanglement is a critical concept in many different fields of physics — in quantum optics, entanglement between a system (i.e. an atom and the electromagnetic field) leads to decoherence, crucial to the understanding of spontaneous emission. In quantum information theory, it is entanglement which is the key ingredient in quantum computing which allows for exponential advantage in computing speed over classical computing for certain algorithms. In high energy physics the entanglement entropy is important to the study of the black hole information paradox, and in condensed matter physics, entanglement plays a key role in the process of thermalization, which bridges a closed-system quantum mechanical approach to thermal equilibrium with the classical one utilizing thermodynamic ensembles. In both quantum and classical mechanics, a fully isolated system will evolve fully deterministically. However, for many systems, in the thermodynamic limit (large system size, long time elapsed since initial condition), the system is assumed to evolve into a thermal state at a certain temperature in which its observables can be calculated probabilistically from a thermodynamic ensemble, with no memory of the initial conditions of the system. In classical mechanics, this apparent discrepancy can be resolved with chaos theory, assuming the system to have after a large enough time explored its phase space ergodically such that initial condition of the system becomes irrelevant. In the language of quantum mechanics, the information encoded in the initial state is remembered by the system at long times, but becomes spread out in a nonlocal fashion as entanglement is generated across the many degrees of freedom of the system. Thus the initial state determines the properties of the thermalized state (temperature), but memory of the initial observables is hidden as it can no longer be accessed by physical local measurements. In classical mechanics, it is typically nonlinear evolution that leads to chaos, while in quantum mechanics the equations of motion are linear, and the properties leading to thermalization are encoded in the eigenstates of the Hamiltonian.

One interesting question in condensed matter physics is how the entanglement in a system is dynamically generated under interactions. If a system of composite subsystems (i.e., a 1D array of spins) is initialized in a separable product state and allowed to undergo general unitary evolution in the presence of interactions, it will generally evolve from a state with zero entanglement to an entangled state. A natural question to ask then is what is the rate at which this entanglement will grow. Furthermore, how can one classify systems based on the characteristics of their entanglement growth?

In this paper, we discuss the results of two papers which interrogate and partially answer these questions, *Ballistic Spreading of Entanglement in a Diffusive Nonintegrable System*[2], and *Quantum Entanglement Growth under Random Unitary Dynamics*[3], published in Physical Review Letters, and Physical Review X, respectively. Both of these papers discuss entanglement growth in a 1D chain of quantum subsystems, starting from an initially unentangled state. The former studies this chain using

full diagonalization of a nonintegrable quantum Hamiltonian giving interactions between the spins, and finds a ballistic (linear in time) growth of entanglement when averaged over random initial conditions. The latter instead uses a discrete quantum circuit model, in which time is discretized and random unitary evolutions are applied to neighbouring "spins" at each time step. In this work, the authors again find ballistic entanglement growth, and map this evolution to a stochastic differential equation used in three different models in classical mechanics, giving a wide variety of perspectives to view the features of entanglement growth. We begin by a discussion of relevant background information and previous findings, and then give a summary of the most important results of both papers.

II. BACKGROUND INFORMATION

Quantum entanglement is a unique quantity in that unlike other conserved quantities which can be transported such as energy and charge, entropy grows and spreads upon interactions with other subsystems in a manner often likened to an epidemic, where information becomes delocalized over a larger volume. Often, entanglement is quantified via the entanglement entropy, or generically, the nth Rényi entropy S_n. For a pure state of an isolated quantum system, the entanglement entropy is zero, but by dividing the system into subsystems (tracing out all but the subsystem of interest), a measure of entanglement of the region considered with the rest of the system can be obtained. Consider a chain of "spin" in a 1D lattice, each with subspace of dimension q. By making a "cut" in the lattice at *bond* location x and tracing out every spin to the left of the cut, dividing it into two regions, one can calculate

$$S_n(x) = \frac{1}{1-n}\log_q(\mathrm{Tr}\rho_x^n), \tag{1}$$

where ρ_x is the reduced density operator for everything right of the cut at x. The most theoretically important Renyi entropy is $\lim_{n\to 1} S_n(x) = S(x)$, where $S(x)$ is the von Neumann entropy:

$$S(x) = -\mathrm{Tr}(\rho_x \log_q \rho_x). \tag{2}$$

In lattice systems, the existence of an upper bound on the propagation of entanglement (generically much less than the speed of light in a vacuum) is rigorously proven, and is known as the *Lieb-Robinson velocity*[4]. However, this result is highly general and it is desirable to have more information on how this information actually propagates in specific systems.

As discussed in the introduction, quantum systems which thermalize do so on the basis of their eigenstates of the system Hamiltonian[5], and these will typically saturate to a state which has entanglement entropy scaling extensively with the volume of the system. However, there also exist systems which do not thermalize on the basis of disorder, instead forming many-body localized states which retain a memory of their initial conditions in local observables and have entanglement entropies that scale with the area of the system in the steady state. These states are known to exhibit logarithmic growth of entanglement entropy over time[6,7].

In quantum systems which do thermalize, *integrable* systems, with extensively many conserved constants of motion, were known prior to the publication of Ref.[2] to have ballistic entropy growth[8]. In these systems, this growth of entropy can be understood from a picture of quasiparticles in the system which propagate ballistically. The work of Ref.[2] of Kim and Huse deals with the growth of entanglement in a *non-integrable* system.

III. BALLISTIC SPREADING OF ENTANGLEMENT IN A DIFFUSIVE NONINTEGRABLE SYSTEM

In this section we summarize the main results and methods of Ref.[2]. In this work, Kim and Huse study a nonintegrable spin chain model in 1D which thermalizes. This model has diffusive energy transfer, which means that the energy transport occurs proportional to $t^{1/2}$. The system consists of a chain of L interacting spin 1/2 particles with transverse and longitudinal fields. The Hamiltonian used to model this system is

$$H = g\sum_{i=1}^{L}\sigma_i^x + h\sum_{i=1}^{L}\sigma_i^z - J(\sigma_1^z + \sigma_L^z) + J\sum_{i=1}^{L-1}\sigma_i^z\sigma_{i+1}^z. \tag{3}$$

The first two terms of Eq. (3) correspond to the transverse and longitudinal fields, respectively. The fourth term is the interaction with open boundary conditions (which are beneficial as periodic boundary conditions reduce the effective chain length, which is already limited due to numerical considerations), and the third is a correction term to reduce the effective longitudinal field at the spin chain boundaries, which is done to reduce finite size effects associated with the boundary conditions. The authors choose parameters $J = 1$, $h = (\sqrt{5}+1)/4$, and $g = (\sqrt{5}+5)/8$, which are chosen to make the system's non-integrability apparent even with finite lattice sizes. The system has only one symmetry (mirroring spins along center point), and the Hamiltonian is diagonalized numerically in each of these parity sectors separately. One way in which the authors can verify that this system is nonintegrable and does not have additional symmetries is by looking at the distribution of the ratio of successive differences in energy eigenvalues[9]. Specifically, if $\lambda_1, \lambda_2, ...$ are the eigenvalues of the Hamiltonian in order from smallest to largest, then one can plot a histogram of the ratios $r_i = (\lambda_{i+2}-\lambda_{i+1})/(\lambda_{i+1}-\lambda_i)$. For an integrable system, one expects this distribution to follow Poisson statistics, whereas for a non-integrable system with no particular symmetries, the probability distribution displays level repulsion, and can be predicted from

random matrix theory. This approach allows the authors to verify their model is nonintegrable, contains no additional symmetries, and (with the aid of looking at the general energy distribution), that the model is sufficiently generic such that the eigenspectrum contains no particular features. This helps the authors conjecture that the results of this model are generalizable to other nonintegrable systems.

The authors consider an entanglement cut in the middle of the spin chain, and parameterize their initial product states as

$$\psi(t=0) = \prod_{i=1}^{L} \left(\cos\left(\frac{\theta_i}{2}\right)\uparrow_i + e^{i\phi} \sin\left(\frac{\theta_i}{2}\right)\downarrow_i \right), \quad (4)$$

where $0 \leq \theta_i < \pi$, and $0 \leq \phi_i < 2\pi$ are random variables. Eq. (4) allows the authors to consider the full space of initial product states. The entanglement entropy used is the von Neumann entropy (Eq. (2)). The authors calculate this entanglement entropy by evolving an initial state to time t for 200 different initial $\{\theta_i, \phi_i\}$ conditions, done separately for each value of t to ensure statistical independence. Ballistic entanglement growth is seen for short timescales until saturation is reached, where the entanglement levels out at a steady state value. Analysis reveals a finite-size scaling form of $S(t) = S_L(\infty)F(t/S_L(\infty))$, where the function $F(x)$ has asymptotic forms $F(x) \sim v_E x$ as $x \to 0$, and $F(x) \sim 1$ as $x \to \infty$, where $v_E \approx 0.7$. Analytic calculations show that the steady state value of entanglement for this model should be $L/2$, or the full length of the chain after the entanglement cut has been made. This allows one to interpret v_E not just as a rate of entanglement growth, but a "speed" of entanglement.

To verify that the model used in Eq. (3) indeed involves diffusive energy transfer, the authors define a "distance" function $R(t)$ which gives the spatial spreading of an energy excitation initially localized at the center bond of the chain away from the center over time. The authors confirm that the spreading of this energy is diffusive (i.e., it has $\sqrt{t}$ time dependence before saturation, and neglecting very short times $t << 1$). The authors conjecture that all local observables for this model (and non-integrable systems in general) should have diffusive behaviour, although they are not able to provide evidence for this.

IV. QUANTUM ENTANGLEMENT GROWTH UNDER RANDOM UNITARY DYNAMICS

In this section we summarize the main results and methods of Ref.[3]. Whereas Kim and Huse studied quantum dynamics using a Hamiltonian and exact diagonalization with random initial conditions, Nahum *et al.* study the dynamics of entanglement by discretizing time evolution and studying the evolution of quantum systems under *random unitary evolution*. The advantage of this approach is that by evolving systems under this "minimally structured" model, the results will presumably be more generic and widely applicable. This work finds again ballistic spreading of entanglement, and maps this growth to the well known Kardar-Parisi-Zhang (KPZ) equation. The KPZ equation has three important applications in classical physics, and each of these can be related to entanglement growth under random unitary evolution. This equation can also be used to deduce critical exponents governing fluctuations in the entanglement growth and correlations, which gives additional insight into entanglement behaviour beyond ballistic growth. Another major finding of this work is that the "speed" of entanglement is generically conjectured to be slower in general than the "operator spreading" of initially localized operators in the Heisenberg picture, in contrast to previously held beliefs.

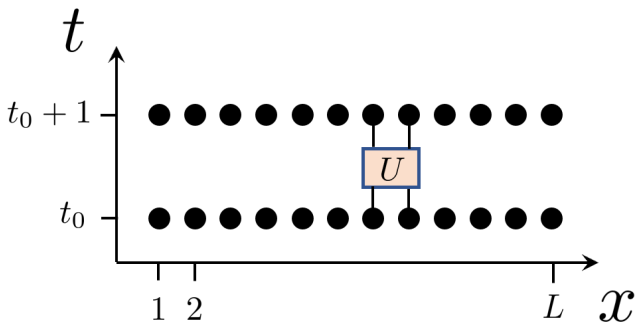

FIG. 1. Model of 1D circuit of L spins, where a random unitary gate couples two randomly selected neighbouring spins at each discrete time step (i.e. from t_0 to $t_0 + 1$), which corresponds to the dynamical rule of Eq. (5) in the $q \to \infty$ limit.

A. Simple 1D Circuit Model

Nahum *et al.* study random unitary circuits in which neighbouring "spins" are able to interact via a random unitary operator in discrete time steps, as shown in Fig 1. Here a simple model is considered where one takes the $n = 0$ Rényi entropy S_0, which simply depends on the number of nonzero eigenvalues of the reduced density operator. This entropy on its own is not always interesting, as it does not quantify how arbitrarily small those eigenvalues may be, but it is shown that in the $q \to \infty$ limit, all Rényi entropies become equal to S_0. In this limit, the entropies (denoting them as S_0, generically) follow the very dynamical rule for timesteps from t to $t+1$:

$$S_0(x, t+1) = \min\{S_0(x-1, t), S_0(x+1, t)\} + 1, \quad (5)$$

where, for each time step, a random *bond* x is chosen and the dynamical rule is applied while enforcing the boundary condition $S_0(x = 1) = S_0(x = L) = 0$. Note that here x labels the L bonds of the lattice, in contrast to Ref.[2]. This rule corresponds to a stochastic surface growth model, which saturates to a pyramid-like shape.

The connection to stochastic surface growth suggests a comparison to the KPZ equation, originally introduced to study stochastic surface growth,

$$\frac{\partial S}{\partial x} = \nu \partial_x^2 S - \frac{\lambda}{2}(\partial_x S)^2 + \eta(x,t) + c, \tag{6}$$

where S denotes the height of a surface at position x, and $\eta(x,t)$ is noise uncorrelated in space and time. The KPZ equation is characterized by critical exponents $\beta = \frac{1}{3}$, governing the size of fluctuations, and $\alpha = \frac{1}{2}$, governing spatial correlations. Specifically, for times prior to saturation, the mean height $h(t)$ grows ballistically with speed v_E with a subleading correction:

$$\langle S(x,t)\rangle = h(t) = v_E t + B t^{\beta}, \tag{7}$$

where B is a constant. The standard deviation of the entropy $w(x,t)$ satisfies

$$w(x,t) = C t^{\beta}, \tag{8}$$

where C is a constant, and the spatial correlation length ξ follows

$$\xi(t) \propto t^{\beta/\alpha}. \tag{9}$$

Nahum *et al.* state that their simulations reveal growth dynamics of the entanglement entropy consistent with the KPZ equation (i.e., it satisfies Eq.'s (7)- (9)), where S denotes the entanglement entropy evolving under Eq. (5).

B. Directed Polymer and Minimal Cuts

A second perspective to view entanglement entropy dynamics by is to consider the "minimal cut". By fixing the top position x, any cut which starts at the bottom of the circuit ($t = 0$, at any location) and ends at x gives a lower bound on the entanglement entropy $S_0(x,t)$ by counting the number of legs the cut passes through. Thus, the best estimate for the entanglement entropy is the cut which passes through the minimal number of legs. It can be shown that all Rényi entropies satisfy $S(x) < S_{\text{cut}}$, where S_{cut} is the number of legs that the curve of the cut passes through. As a consequence of this, the "best" (lowest) bound for the entanglement entropy is given by the minimal cut which passes through the smallest amount of legs of the circuit. In a random circuit like the ones described in this work, the "geometry" of the circuit will be random, which leads one to draw analogy to the problem of finding an energy minimizing shape in a disordered medium. This "directed polymer" problem can be mapped to the KPZ equation by moving to a continuum description of the relevant coordinates, and thus shares the same critical exponents and universal properties. The authors also show that the scaling forms describing this model are equivalent to those in the work by Kim and Huse[2], described in Section III.

The minimal cut picture is only a lower bound for the von Neumann entropy (except in the $q \to \infty$ limit, where it gives it exactly), however the authors conjecture that the properties of entanglement entropies that can be calculated with this approach share universal features with the actual dynamics. One major advantage of the minimal cut picture is that unlike the other pictures presented in this work, the minimal cut picture can be generalized to higher dimensions. In this manner, the 1D polymer is replaced with a d dimensional membrane embedded in $d+1$ spacetime, which is pinned by disorder, and the "minimal membrane" gives an estimate of the entanglement entropy.

C. Operator Spreading and Hydrodynamics

The last of the three models that follow KPZ dynamics that can be related to entanglement growth in random unitary circuits relates to operator growth and hydrodynamics. An operator $\mathcal{O}_i(t)$ that acts locally on a specific spin i at $t = 0$ will in general grow ballistically – i.e., the number of spins it acts on over time in the Heisenberg picture will grow linearly with a certain speed v_B. One of the main results of the work by Nahum *at al.* is the show that this speed v_B is in general larger than the speed at which entanglement actually grows. The entanglement growth speed is instead a function of the collective behaviour of operator spreading, which can be mapped again to KPZ dynamics by an analogy to stochastic hydrodynamics. To study the growth of these operators, the authors study Clifford circuits, which give rise to a restricted set of possible unitary evolutions. Specifically, Clifford circuits consist of unitary gates which take (tensor products of) Pauli matrices to other Pauli matrices under the transformation $\mathcal{O}_i \to U(t)\mathcal{O}_i U^{\dagger}(t)$. Clifford circuits do not study quantum evolution in full generality, but have the advantage of being classically simulable in polynomial time, in contrast to general quantum evolution.

The language of operator spreading and entanglement growth is best described using the "stabilizer" formalism. Restricting themselves to Clifford evolution, the authors allow the initial state of L qubits ($q = 2$) in a 1D chain to be a product state of spins aligned in the X, Y, or Z directions, such that the initial state $\psi(0)$ can be described with L Pauli operators $\{\mathcal{O}_i\}$ for which $\psi(0)$ is an eigenstate with eigenvalue 1:

$$\mathcal{O}_i \psi(0) = \psi(0). \tag{10}$$

In connection to group theory, they denote these L operators the *stabilizers* of $\psi(0)$. Note that the Pauli matrices X, Y, Z, are not all independent; for example, $Y = -iXZ$. Furthermore, the overall sign of the stabilizers is not relevant and does not need to be kept track of as it is not involved in the unitary evolution and does not play a role in the entanglement properties. Thus, any

stabilizer can be represented as a string

$$O_i(t) = X_1^{x_{1,i}} Z_1^{z_{1,i}} ... X_L^{x_{L,i}} Z_L^{z_{L,i}}, \tag{11}$$

where the $L \times L$ matrices $x_{i,j}$, $z_{i,j}$ have binary entries which fully determine the state $\psi(t)$. For Clifford circuits, the unitary gates (evolutions that couple two qubits) can be selected from the generating set of Hadamard, phase, and CNOT gates, for which the entries of $x_{i,j}$ and $z_{i,j}$ can be updated efficiently. Furthermore, the entanglement entropy (for Clifford circuits, all Rényi entropies are equivalent) can be calculated from these matrices.

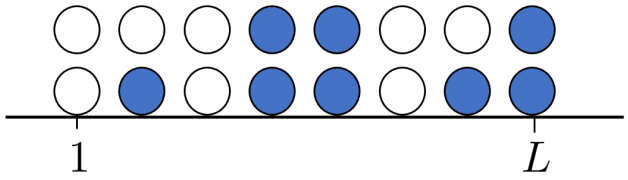

FIG. 2. Schematic of the operator spreading picture, where white circles represent left endpoints of a stabilizer and blue circles denote right endpoints, for L spins. In general, the right "particles" will drive towards the right over time, and the white particles will drift toward the left. An initially unentangled state corresponds to equal densities of white and blue particles.

The stabilizer formalism allows for qualitative insight into how entanglement growth can be related to the spreading of operators. A stabilizer at $t = 0$ for a state initially unentangled will be completely localized at a spin at location i on the 1D lattice. As time increases, as previously mentioned, the rate at which the stabilizer increases its length (by acting on more spins) is ballistic and can be ascribed "butterfly speed" (in relation to the butterfly effect of chaos theory) v_B. One can create a picture of how this functions by labelling the left and right endpoints of the ith stabilizer by l_i and r_i, respectively. By analogy, these left and right endpoints can be seen as two types of fictitious particle, represented by white and blue circles as in Fig 2. Noting that the L stabilizers are not necessarily unique descriptions of the state at a given time (for example, two stabilizers can be multiplied to get another stabilizer), the authors use a "clipped gauge" in which they enforce the criterion that the total amount of l and r particles at a location i must be equal to 2. Enforcing this requirement during unitary evolution ensures that the stabilizers remain independent. In this representation, it is easy to calculate the entanglement entropy based on the distribution of these particles. Specifically, the entanglement entropy $S(x)$ at a cut x is equal to the number of r particles right of the cut minus the number of sites right of the cut. The dynamics in this model can be understood then as r particles spreading to the right over time with the restriction that more than two r particles can not exist at a certain location at a time, described as a "traffic jam" phenomenon. From this framework, the authors can explicitly derive the KPZ equation for Clifford circuits.

The "clipping" that is done to enforce the criterion described above ensures that the stabilizers remain independent, but it also effectively slows their growth. This is a key hint to understanding why, in general, the entanglement speed v_E is smaller than the butterfly speed v_B. The latter describe the growth speed of an operator considered in isolation, but by enforcing an independence requirement on the operators which describe the quantum state as a whole, it can be seen that the *collective* speed of operator growth, which corresponds to the speed of entanglement, is slower. The authors summarize this principle with the statement that "thermalization is slower than operator spreading".

D. Numerical Evidence of KPZ Behaviour

Finally, the authors provide numerical evidence for Clifford circuits (and more generic quantum circuits) that the behaviour of random unitary evolution does in fact give rise to KPZ behaviour, and we summarize some of this here. For Clifford circuits, they randomly select for each bond in the circuit at each time step from either the identity gate, a left-CNOT gate, or a right-CNOT gate. Starting from initial conditions of all spins polarized in y-direction, they calculate the von Neumann entropy $S(x)$, showing again the pyramidal shape that the model in Section IV A results in.

Next, the critical exponents of the KPZ equation can be verified by averaging over many different iterations and plotting $h(t)$ and $w(t)$, as defined in Eq.'s (7) and (8). From numerical fits, the authors find exponents of $\beta = 0.33 \pm 0.01$ for the fit of $h(t)$, and $\beta = 0.32 \pm 0.02$ for the plot of $w(t)$, including a subleading correction, in excellent agreement with the KPZ values. They also find by considering spatial correlations that the KPZ value of $\alpha = 1/2$ is well replicated for short times prior to saturation.

By comparing the entanglement growth speed as defined by Eq. (7) with the butterfly speed given by the growth of a single Pauli string, the authors find $v_E = v_B/2$, which agrees with exact calculations in the analytically solvable $q \to \infty$ model.

Going beyond Clifford circuits, the authors also do simulations with "phase evolution" and "universal evolution" using matrix product states. Phase evolution corresponds to applying, in addition to the random two-qubit gates used in the Clifford circuits, one-qubit gates that apply a random phase rotation about the z axis to each qubit, while universal evolution includes this plus a Hadamard gate (which allows the full space of unitary evolution to be sampled). These calculations are much more computationally intensive, and as such the limited simulation times that can be accessed means the authors can not extract from curve fitting critical exponents; however, they do plot curve fits using the KPZ

exponents as fixed parameters which do agree with the data. This, coupled with the heuristic pictures and arguments given in the rest of the paper, lead the authors to conjecture that the KPZ behaviour is universal for generic (random) unitary evolution of quantum systems.

V. CONCLUSIONS

In conclusion, the papers we have discussed have provided much evidence that in general, nonintegrable quantum systems in 1D have entanglement entropy which grows ballistically in time from an initially unentangled state. Kim and Huse (Ref.[2]) showed this by taking a specific nonintegrable model of a spin 1/2 chain in 1D which has diffusive energy transfer, and exactly diagonalizing the Hamiltonian, conjecturing that this ballistic entanglement growth would apply to any generic nonintegrable model with diffusive local observables. Nahum *et al.* (Ref.[3]), in contrast, considered random unitary circuits, which have minimal structure and thus likely provide a better estimate of how generic quantum systems will behave. Nahum *et al.* provided evidence that random unitary circuits will exhibit behaviour in the KPZ universality class, by relating three problems in classical physics governed by the KPZ equation to entanglement growth, and showing how this can be generalized to higher dimensions using a "minimal membrane" picture, generalizing the directed polymer model in a disordered medium. They also found, notably, that the rate of entanglement growth is in fact a speed, and that this speed is generically slower than the speed of operator spreading; "thermalization is slower than operator spreading". They provided an explanation of this finding by using Clifford circuits as an example, where requiring the growth of stabilizers into a certain region (directly correlated with entanglement) is slowed relative to generic operator spreading due to the "clipping" requirement of stabilizers to keep them independent of each other. To give an example of an interesting finding in this topic after the publication of these papers, in 2019, Rakovszky *et al.*[10] found using both spin chain diagonalization and random unitary circuits that for nonintegrable systems with a conserved quantity (e.g., energy) that spreads diffusively, the Rényi entropies for $n \geq 2$ grow diffusively. This is interesting as they point out for large subsystems that only these entropies are currently experimentally accessible.

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