The disordered Floquet Ising chain

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The aim of this report is to review the phase diagram of the periodically-driven (a.k.a. Floquet) disordered Ising chain, as obtained in Ref. [1]. This model is important as it is one of the simplest models that display spontaneous discrete time-translation symmetry breaking (dTTSb). I will also review Ref. [2], where the phase transitions of this model are studied in terms of emergent Majorana fermions. Finally, I discuss the stability of the one of the phases even in the absence of Ising symmetry following Ref. [3].

I. INTRODUCTION

Traditional phases of matter were defined using Landau symmetry-breaking theory [4, 5] in undriven local systems as follows. First, we identify the symmetries of the Hamiltonian and an local observable that transforms non-trivially under said symmetries, a.k.a. order parameters. We then say that a symmetry is broken if the corresponding order parameter O_i display long-range order: $\langle O_i O_j\rangle - \langle O_i\rangle\langle O_j\rangle \xrightarrow{|i-j|\to\infty} C^2 \neq 0$.

After years of studying phases of matter in time-independent systems there was an obvious question: what new phenomena can appear in time dependent systems. In particular, how do we define a phase of matter in this setting? A natural place to look for new physics is to study some version of the quantum Ising chain as it lends itself to analytical and numerical techniques. The particular model I will explain is the periodically-driven disordered Ising chain. The interest on this model will hopefully becomes clear in the following.

First of all, the period-driven systems, a.k.a. Floquet systems [6, 7], posses a discrete time translation symmetry that can in principle be spontaneously broken [8]. There is no straightforward way to use the standard Landau symmetry-breaking theory [4, 5] as we can no longer define a thermal state because energy is no longer well-defined. Therefore we need to study the properties of correlation functions in generic states of the Hilbert space but it is often good enough to restrict to study all the eigenstates. This notion is called eigenstate order [9–12] and will be explained in Section III. It is common believe [13, 14] that translation invariant interacting systems will "heat to infinite temperature" under a periodic drive, i.e. all correlation functions show no dependence on the initial state for long times, which is already a possibility in static systems. The intuition from a linear response point of view is that in the presence of translation invariance, the eigenstates are extended and can then easily exchange energy between each other. The role of disorder will be to localize the modes in the Ising chain even in the presence of small interactions [11, 15], thus avoiding all-to-all energy exchange and heating.

In the following, I will review some background notions of periodically-driven (a.k.a. Floquet) systems, discrete time-translation symmetry breaking (dTTSb), eigenstate order and the (static) disordered Ising chain. After this I will review the phase diagram of the disordered Floquet Ising chain in the presence of Ising symmetry.

II. BACKGROUND: FLOQUET SYSTEMS AND DTTSB

Floquet systems [6, 7] have a periodic Hamiltonian, $H(t) = H(t+T)$ for some $T > 0$. The Floquet unitary is defined as

$$U_F = U(0,T) = \mathcal{T}\exp\left[-i\int_0^T H(t)\,\mathrm{d}t\right],$$

where $\mathcal{T}$ is time ordering. U_F corresponds to time evolution for a period T. Instead of diagonalizing the Hamiltonian at every time, it is convinient to diagonalize the Floquet unitary. The Floquet eigenvectors $|\phi_\alpha\rangle$ with quasi-energy ε_α are defined as

$$U_F\,|\phi_\alpha\rangle = e^{-iT\varepsilon_\alpha}\,|\phi_\alpha\rangle\,,$$

with quasi-energies defined modulo $2\pi/T$. Studying the properties of these eigenvectors is enough to understand the dynamics of the system under discrete time evolution with time-step T as they form a basis for the state space.

As in undriven systems, we can think of $|\phi_\alpha\rangle$ as being eigenstates of an effective Floquet Hamiltonian H_F defined by $U_F = \exp(-iTH_F)$. The issue with this definition is that H_F is not uniquely defined due to the periodic identification of the quasi-energies and, in addition, there is no guarantee that H_F has nice properties as locality. If H_F was local, we could use all the machinery already known for equilibrium systems. Nevertheless, this is not always true. An special case where H_F is easily constructed is when T is much smaller that the energy scales of $H(t)$ at everytime. In this case, we could just use perturbation theory to find a local H_F.

As the Hamiltonian has period T, there is a discrete time-translation symmetry of shifting time by integer multiples of T. Similar to the standard spontaneous breaking of a symmetry, a Floquet system has spontaneous dTTSB [16] when the correlation function in the infinite system size limit of local operators whose distance is taken to infinity have a period larger than T for late times.

III. BACKGROUND: EIGENSTATE ORDER AND PHASES OF THE DISORDERED ISING CHAIN

The traditional study of static phases of matter centered in the properties of observable restricted on the ground-state manifold or thermal states. Nevertheless, while studying many-body localized (MBL) systems [9–12] it became clear that we should also consider other properties of the energies and eigenstates of the Hamiltonian, thus referred to as eigenstate order. Among these properties are the entanglement structure of the eigenstates and spectral properties of the energies (e.g. distribution of the difference between consecutive energies).

For concreteness, consider the disordered quantum Ising chain consisting of spin-1/2 degrees of freedom on an open chain of size L with Hamiltonian

$$\begin{aligned} H = &\sum_{i=1}^{L-1} J_i \sigma_i^z \sigma_{i+1}^z + \sum_{i=1}^{L} h_i \sigma_i^x \\ &+ K_x \sum_{i=1}^{L-1} \sigma_i^x \sigma_{i+1}^x + K_z \sum_{i=2}^{L-1} \sigma_{i-1}^z \sigma_{i+1}^z. \end{aligned}$$

where $\sigma_i^{a=x,y,z}$ are Pauli matrices of the spin on site i. J_i and h_i correspond to random interactions and magnetic fields respectively. K_x and K_z are small interaction terms. As pointed out in [9], these model has (among other phases) an MBL paramagnet (PM) phase and a MBL spin-glass (SG) phase[17]. The eigenstates deep in the MBL PM phase correspond to eigenstates tensor products of spins polarized in the transverse direction. The MBL SG phase corresponds to the Ising dual of the PM, i.e. the domain walls (change in orientation of spins along the interaction direction) are the ones that are frozen. The main physical difference between these two phases is the presence long-range magnetic order in the SG phase, i.e. all eigenstates in SG break the Ising symmetry.

Away from deep of the phases but still in the localized regime, the phases can be understood in terms of an emergent integrability [18, 19]. The upshot is that in this regime, there is a change of basis from the original spin dofs σ_i^a to some new dressed versions of them τ_i^a that are exponentially localized at site i and commute with each other. The new variables are called l-bits (see Fig. 1). The emergent integrability means that in the infinite size limit, the Hamiltonian commutes with an infinite number of conserved charge that correspond to an infinite set of the l-bits.

In particular in the MBL PM and MBL SG, the Hamiltonians written in l-bits basis become respectively

$$\begin{aligned} \tilde{H}_{PM}[\{\tau_i^x\}] &= \sum_i \tilde{h}_i \tau_i^x + \sum_{ij} \tilde{K}_{ij} \tau_i^x \tau_j^x + \dots \\ \tilde{H}_{SG}[\{\tau_i^z \tau_{i+1}^z\}] &= \sum_{ij} \tilde{J}_{ij} \tau_i^z \tau_j^z \\ &\quad + \sum_{ijkl} \tilde{L}_{ijkl} \tau_i^z \tau_j^z \tau_k^z \tau_l^z + \dots \end{aligned} \tag{1}$$

where $\tilde{h}_i, \tilde{K}_{ij}, \tilde{J}_{ij}$ and $\tilde{L}_{ijkl}$ are exponentially decaying couplings and the dots correspond to terms with more spins. In the MBL PM the conserved charges are $\{\tau_i^x\}$ while in the MBL SG are $\{\tau_i^z\}$. As the SG Hamilto-

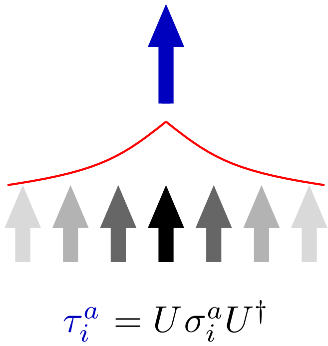

FIG. 1. (Adapted from Ref. [16], Source: Vladimir Calvera) The l-bits τ_i^a are dressed versions of the physical σ_i^a with exponentially suppressed (in distance) weight on the other spins.

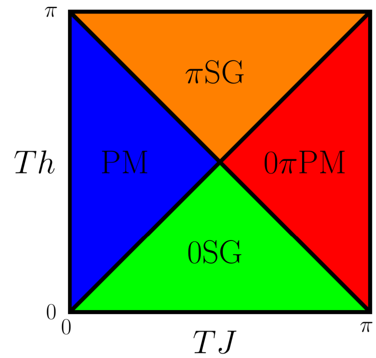

FIG. 2. Phase diagram for the Floquet Ising chain adapted from Ref. [1] (Source: Vladimir Calvera).

nian only depends on the product of two τ_i^z, we see that the spectrum will be degenerated as there is an emergent symmetry $\tilde{P}_x = \prod_i \tau_i^x$ which corresponds to flipping the value of the l-bits in the z basis. This is true in the infinite system size, while for the finite system the degeneracy is only true up to exponential accuracy and the different between the two states correspond to filling a mode localized on the boundaries of the chain.

The notion of eigenstate order is extended to the Floquet setting [1] by studying the properties of the Floquet eigenstates evolved over a period instead of the energy eigenstate of the static case.

IV. MAJORANA LANGUAGE

It is easier to understand the degeneracy of both the undriven and driven systems by making a change of basis to Majorana fermions. For a more complete review see Appendix A. The upshot is that there are two Majorana fermions $\gamma_i^{A,B}$ build as $\sigma_i^{z,y}$ dressed by a chain of all σ_j^x to the left of i. These operators are Majorana fermions because they anticommute with each other and square to one. The zero mode of the previous section can be understood as the existence of linear combinations of the Majorana modes, $\gamma_{L/R}$, that commute with H and are localized on the left/right edge.

V. THE MODEL

Similarly to the static disorder chain of the previous section, the Floquet disorder chain consists of an open chain of L spin-$\frac{1}{2}$ degrees of freedom driven by a Hamiltonian that for a half the period equals a Hamiltonian deep in the SG phase (H_z) and the other half of period is a Hamiltonian deep in the PM phase (H_x), i.e.

$$H(t) = \begin{cases} H_z\ , & t \in [0, T/2) \\ H_x\ , & t \in [T/2, T) \end{cases},$$
$$H_z = \sum_{i=1}^{L-1} J_i \sigma_i^z \sigma_{i+1}^z + K_z \sum_{i=2}^{L-1} \sigma_{i-1}^z \sigma_{i+1}^z,$$
$$H_x = \sum_{i=1}^{L} h_i \sigma_i^x \ + K_x \sum_{i=1}^{L-1} \sigma_i^x \sigma_{i+1}^x,$$

with $h_i(J_i)$ taken from uniform distributions with mean $h(J)$ and width $\delta_h(\delta_J)$. K_x and K_z correspond to small interactions terms. Important to the study of phases are the Ising symmetry ($P_x = \prod_{i=1}^{L} \sigma_i^x$) and the dTTS with period T.

VI. PHASE DIAGRAM

Before proceeding, it is worth pointing out some relations between different values of h and J. Consider the unitary transformations $P_z = \left(\prod_{j=1}^{L} \sigma_j^z\right)$ and $Q_x = \left(\prod_{j=1}^{\lfloor L/2 \rfloor} \sigma_{2j}^x\right)$, then

$$P_z^{-1} \cdot U_F(J_i, h_i) \cdot P_z = U_F(J_i, -h_i), \tag{2}$$
$$Q_x^{-1} \cdot U_F(J_i, h_i) \cdot Q_x = U_F(-J_i, h_i). \tag{3}$$

Which means that the phases with (J, h), $(-J, h)$, $(J, -h)$ and $(-J, -h)$ are all easly related.

There is also a periodicity in J and h. Consider

$$\begin{aligned} h_i &\longrightarrow h_i + \pi/T: \quad U_F \longrightarrow P_x U_F (-i)^L, \\ J_i &\longrightarrow J_i + \pi/T: \quad U_F \longrightarrow U_F \sigma_1^z \sigma_L^z (-i)^{L-1}. \end{aligned} \tag{4}$$

As global phases only shifts the whole spectrum and Floquet eigenstates are unchanged, the physics is invariant under global shifts by $2\pi/T$ of all J_i's or h_i's. Combining the two previous relations, it is enough to study the (J, h) restricted to $[0, \pi/T] \times [0, \pi/T]$.

The phases were h and J are both small compared to π/T, the phases will correspond to the undriven phases as we can do perturbation theory. In particular for $h \ll J$, the Floquet eigenstates will look like eigenstates of H_z and thus, this corresponds to the MBL SG phase. Similarly, for $J \ll h$, the Hamiltonian describes a PM phase. To understand the phase when h or J are large, i.e. close to π/T, we can use the relations in Ref. 4. For h close to π/T, the π/T shift maps the problem to that of small h but paying the price of an extra P_x. The Floquet eigenstates are thus eigenstates of P_x and the PM Floquet operator: the states odd under P_x acquire an π/T shift in their quasi-energies. This will mean that in finite size, the quasi-energies will come in pairs such that their difference is π/T up to exponential (in L) accuracy. Finally, consider J close to π/T so that the shift implies that U_F now corresponds to the PM phase up to the boundary terms $\sigma_1^z \sigma_L^z$. In this case, to leading order we can factor U_F in a bulk term and a boundary term of the form $\sigma_1^z \sigma_L^z \epsilon^{-i\frac{T}{2}(h_1 \sigma_1^x + h_L \sigma_L^x)}$ with eigenvalues $\{+1, +1, -1, -1\}$. The last point implies that quasienergies will come in quadruplets $\{\varepsilon, \varepsilon, \varepsilon + \pi, \varepsilon + \pi\}$. This is only true up to exponential accuracy for a finite system.

Even though I have only argued the phase diagram close to the boundaries, more careful studies show that these are only phases (first obtained in Ref. [1]) which can be understood from symmetry considerations show a simple phase diagram as shown in Fig.2 (See Ref. [2] or App. A). The PM and 0-SG are essentially the same as the MBL PM and SG phases of the undriven disordered Ising chain. The new phases are the 0π-Paramagnet (0π-PM) and the π-Spin glass (π-SG). Both phases display spontaneous dTTSB, while only the eigenstates of later also show long-range order. The 0 or π in front of PM and SG denote quasi-degeneracies in the spectrum of quasi-energies mentioned in the previous paragraph.

Away from the phase boundaries, the phases can be understood in terms of l-bits similarly to the undriven case. The main difference to the undriven case is that in the novel phases the Floquet operators have the l-bit version of the extra boundary terms discussed previously. This can be summarized by

$$\begin{aligned} U_F^{(PM)} &\sim e^{-iT\tilde{H}_{PM}} \\ U_F^{(0SG)} &\sim e^{-iT\tilde{H}_{SG}} \\ U_F^{(\pi SG)} &\sim \tilde{P}_x e^{-iT\tilde{H}_{SG}} \\ U_F^{0\pi PM} &\sim \tau_1^z \tau_L^z e^{-iH_{PM}} \end{aligned} \tag{5}$$

where $\tilde{P}_x = \prod_i \tau_i^x$.

VII. PHASE TRANSITIONS

Following Ref. [2], the different phase transitions of the non-interacting case can be understood as a infinite randomness fixed point (IRFP) of underlying Majorana chains at 0 and π/T quasienergy. In order to explain what this means, let's start with the undriven case without interactions.

The IRFP is obtained by considering a strong disorder real space renormalization. This renormalization group (RG) procedure is more easily understood in the Majorana picture as h_i and J_i can both be understood as hopping terms t_j. The idea of the procedure is that at each step, there is a j_1 such that $t_{j_1}\gamma_{j_1}\gamma_{j_1+1}$ is the largest term. The RG procedure then consists of 'integrating out' γ_{j_1} and γ_{j_1+1} so that there are new effective couplings for the remaining DOFs. The final result is that we have Majorana dimers with different lengths and strengths. Depending on $J \lesseqgtr h$, most of the dimers will be between Majoranas of the same side of the unit cell leading to the possible presence of zero modes. In the case $h = J$, there is no clear winner between the two kinds of bonds and Ref. [20] showed that in this case the distribution of effective couplings increase without bond but with zero mean. This is the IRFP and has a characteristic critical exponent involving the golden ratio. An alternative way to understand this transition is by noticing that when $J \approx h$, there are going to be regions where the largest couplings are bonds and other where the magnetic fields are largest. By integrating out the couplings inside this regions, there will be an effective Hamiltonian for the remaining degrees of freedom. The transition of the model can then be understood by the criticality of this emerging chain of modes at zero energy.

A similar picture still holds in the Floquet system but should also account couplings flowing towards π/T. An important parameter will then be the fraction of couplings (including J and h) near ('flowing towards') 0, n_0. It is also convenient to define the analogous parameter of π/T, $n_\pi = 1 - n_0$. The know phases PM and $0SG$ correspond to $n_0 > n_\pi$, while the new phases correspond to $n_\pi > n_0$. The transition within the regions $n_\pi \lesseqgtr n_0$ are tuned by the asymmetry between the J_i and h_i distributions as these dictate whether the end chain will be mostly condensed domain walls or frozen spins. The qualitatively new transitions appear when $n_0 = n_\pi = \frac{1}{2}$. Consider first points away from the multicritical point ($Th = TJ = \pi/2$). There are going to be regions where most of the couplings flow towards 0 and others towards π/T. On the boundary of these regions there are going to be modes at π/T-quasienergy either from the $PM - \pi SG$ boundaries or the $0SG - o\pi PM$ boundary (where the zero mode from each region cancel each other $(\sigma^x)^2 = \mathbb{1}$). These boundaries are called $0\pi - DW$ in the language of Ref. [2]. The transitions can then be understood as the phase transition of the emerging chain at quasienergy π/T. Finally, the multi-critical point corresponds to the simultaneous criticality of both emerging chains. The results were confirmed in Ref. [2] by calculating the effective central charge by studying the entanglement entropy at the phase transitions.

Interactions seem to not modify the phase transition universality class and can be argued by the irrelevance under RG of short-range interactions in the emergent chains.

VIII. ABSOLUTE STABILITY OF π-SPIN GLASS

A natural question is whether the identified phases are stable if we break the Ising symmetry. Said in other words, can we go from one phase to another without going through a phase transition once we allow Ising symmetry breaking perturbations? Ref. [3] and references therein argued (analytically and with numerical evidence) that only the πSG phase is stable.

The analytical argument was carried using the l-bits picture previously mentioned: weak perturbations are expected to be able to be 'factored' by a unitary such that there is an emergent Ising symmetry as the system size goes to infinity[21].

For the numerical check, there are two things that were proved in Ref. [3]. First, they checked the robustness of the π-pairing in the spectrum, i.e. the symmetry of the quasi-spectrum under π/T-shifts. This is done by perturbing the πSG phase by considering

$$\bar{U}_F = P_x \exp[-iTH_z - i\lambda V] \tag{6}$$

$$V = \sum_{i=1}^{L} \sum_{a\in\{x,y,z\}} h_i^a \sigma^a \tag{7}$$

with $T\bar{J}_i = 1$, $T\delta J_i = 0.5$ and the symmetry breaking fields h_i^a taken from uniform distributions but with different parameters for each $a \in \{x, y, z\}$ in order to break all the symmetries of the unitary. The spectral gaps are defined by considering the level spacings $\Delta_0^i = \varepsilon^{i+1} - \varepsilon^i$ and $\Delta_\pi^i = \varepsilon^{i+N/2} - \varepsilon^{i+N/2} - \pi/T$, where N is the Hilbert space dimensions. Then the 0 and π spectral gaps are defined as the log-average over i and disorder of Δ_0^i and Δ_π^i. Ref. [3] found that for all proved system sizes ($L \in \{6, 8, 10, 12\}$) there is a finite region of λ where $\Delta_\pi \ll \Delta_0$ (where the ration is of order $\mathcal{O}(10^{-10})$).

The second numeric probe is the dynamics of Floquet operators under time evolution in T steps. Specifically, the operators $\sigma^{x,y,z}$ expectations value in product states of spins pointing randomly on the Bloch sphere show two large peaks at 0 and π/T frequencies and other smaller peaks due to the glassiness of the phase that are expected to decay in the large L and $t = nT$ limits.

IX. FINAL REMARKS

I have review different aspects of the Ising chain driven by two competing Hamiltonians that ultimately give rise to two new phases not possible in equilibrium. The main new ingredient seems to be that allowing time-periodic Hamiltonians allow to introduce 'symmetry defects' along the time direction, that in the studied model correspond to either the global P_x symmetry or the localized Majorana modes. This is an idea that can be generalized using the mathematical techniques to classify equilibrium topological phases. In addition to this, there are other models with a global symmetry G different from $\mathbb{Z}_2$ (Ising) that show stability under small G-symmetry breaking perturbations (e.g. see Ref. [22]).

Appendix A: Jordan-Wigner Transformation

The Jordan-Wigner (JW) transformations is a change of variables between a spin $\frac{1}{2}$ system and Majorana fermions in $1D$. This is obtained by defining

$$\gamma_{i,A} = \left(\prod_{j<i} \sigma_j^x\right)(+\sigma_i^z) \tag{A1}$$

$$\gamma_{i,B} = \left(\prod_{j<i} \sigma_j^x\right)(+\sigma_i^y). \tag{A2}$$

so that $\gamma_{I=(i,X)}$ satisfy fermionic canonical commutation relations

$$\{\gamma_I, \gamma_J\} = 2\delta_{IJ}. \tag{A3}$$

When dealing with periodic boundary conditions there is a caveat at the boundary as $\gamma_{L+1} \neq \gamma_1$, but $\gamma_{L+1} = \left(\prod_j \sigma_j^x\right)\gamma_1$.

Some useful expressions are $(i < j)$

$$\begin{aligned} \sigma_i^x &= i\gamma_{iA}\gamma_{iB}, \\ \sigma_i^z\sigma_j^z &= \gamma_{i,A}\left(\prod_{i\le k<j} \sigma_k^x\right)\gamma_{jA}, \\ \sigma_i^z\sigma_{i+1}^z &= i\gamma_{i,B}\gamma_{i+1,A}. \end{aligned} \tag{A4}$$

1. Hamiltonian in Majorana basis

The most general Hamiltonian being considered is

$$\begin{aligned} H = &\sum_{i=1}^{L-1} J_i\sigma_i^z\sigma_{i+1}^z + \sum_{i=1}^{L} h_i\sigma_i^x \\ &+ K_x\sum_{i=1}^{L-1}\sigma_i^x\sigma_{i+1}^x + K_z\sum_{i=2}^{L-1}\sigma_{i-1}^z\sigma_{i+1}^z. \end{aligned} \tag{A5}$$

Using the expressions from the previous section

$$\begin{aligned} H = &\sum_{i=1}^{L-1} iJ_i\gamma_{iB}\gamma_{i+1A} + \sum_{i=1}^{L} h_i i\gamma_{iA}\gamma_{iB} \\ &- K_z\sum_{i=1}^{L-1}\gamma_{iA}\gamma_{iB}\gamma_{i+1A}\gamma_{i+1B} \\ &- K_x\sum_{i=2}^{L-1}\gamma_{i-1B}\gamma_{iA}\gamma_{iB}\gamma_{i+1A} \end{aligned} \tag{A6}$$

where $\gamma_{i,A/B}$ are Majorana operators localized at site i with an index A/B that can be though of as a left/right sublattice index. Written in this way, it is clear that J_i and h_i play the role of hopping amplitudes, with the former being inter-cell hoppings and the later being intra-cell hoppings. The terms in the second line are interactions.

2. Ising Duality

Forgetting about subtleties coming from the boundary (or in an infinite open system), we can see that we can perform an inverse Jordan-Wigner transformation that combines Majoranas from different unit cells $(\gamma_{iB}, \gamma_{i+1A} \sim \tilde{\gamma}_{i+\frac{1}{2}A}, \tilde{\gamma}_{i+\frac{1}{2}B})$ to write

$$\begin{aligned} H = &\sum_{i=1}^{L-1} J_i\tilde{\sigma}_{i+\frac{1}{2}}^x + \sum_{i=1}^{L} h_i\tilde{\sigma}_{i-\frac{1}{2}}^z\tilde{\sigma}_{i+\frac{1}{2}}^z \\ &+K_x\sum_{i=1}^{L-1}\tilde{\sigma}_{i-\frac{1}{2}}^z\tilde{\sigma}_{i+\frac{3}{2}}^z \\ &+ K_z\sum_{i=1}^{L-2}\tilde{\sigma}_{i+\frac{1}{2}}^x\tilde{\sigma}_{i+\frac{3}{2}}^x. \end{aligned} \tag{A7}$$

where we implicitly introduce new majorana modes ($\gamma_{0,B}$ and $\gamma_{L+1,A}$) that do not appear in the Hamiltonian but are needed to define the spin operators.

Therefore, the original Ising chain is dual (up to some the extra modes) to another Ising chain defined on the dual lattice where $J_i \leftrightarrow h_i$ and $K_z \leftrightarrow K_x$.

Applying this duality to the model we studied, we get

$$U_F = U_xU_z = \tilde{U}_z\tilde{U}_x = \tilde{U}_x^{-1}\cdot\tilde{U}_F\cdot\tilde{U}_x. \tag{A8}$$

Therefore, we establish a symmetry of Fig. 2 under a reflection in the $Th = TJ$ line: The phases mirrored under this line should have a relative zero mode and break or not break the Ising symmetry.

3. 0π-Ising duality

As U_F and $U_F^\dagger$ have the same eigenvectors and opposite spectrum, we can relate the phase of U_F with that of $U_F^\dagger$. I can write

$$U_F^\dagger = U_z^{-1}\cdot U_x^{-1} = \tilde{U}_x^{-1}\cdot\tilde{U}_z^{-1}, \tag{A9}$$

so that $(J,h) \sim (-h,-J)$ up to the extra zero modes [23]. By combining this with the $(J,h) \longrightarrow (J+\pi/T, h+\pi/T)$

transformation of the main text, $(J, h) \sim (\pi/T - h, \pi/T - J)$ up to a mode with π/T quasienergy (the zero mode is cancelled). This gives us another symmetry of Fig. 2 under reflection of the line $J + h = \pi/T$.

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